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Report R-9

THEORETICAL BACKGROUND OF INERTIAL NAVIGATION FOR SUBMARINES

INSTRUMENTATION LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Cambridge 39, Mass.

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Report R-9

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THEORETICAL BACKGROUND OF INERTIAL NAVIGATION FOR SUBMARINES.

March 1951

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(10) Forrest E. /Houston
John /Hovorka

INSTRUMENTATION LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Prepared for Publication by Jackson and Moreland

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PREFACE

The problem of guiding a submerged submarine requires the use of essentially new navigation techniques. One such method involves the application of completely self-contained inertial systems such as are being developed for long-range all-weather aircraft operations. Accordingly, Instrumentation Laboratory of Massachusetts Institute of Technology has undertaken the study of the application of inertial techniques to the navigation of submarines. This study is conducted under contract N5ori-07850 with the Office of Naval Research.

Two reports to O.N.R. on this study will treat each of the two major phases of the work.

a. "Theoretical Background of Inertial Navigation for Submarines", which considers the physical background of inertial navigation and discusses idealized systems for indication of position, with no conclusions drawn as to the relative suitability of the different systems for submarine operation.

b. "Characteristics of Systems Feasible for Inertial Navigation of Submarines", which considers the effect of imperfections in actual instrumentation, e.g., "noise" and "drift", on system performance, leading to recommendation of one inertial system as most suitable for submarine operation.

Particular mention should be made of the work of Mr. Forrest E. Houston for general supervision of the material contained in the report, of Mr. John Hovorka for the actual writing of the report, and the Technical Publications Division of Jackson & Moreland for the preparation of the report.

Cambridge, Mass.
March, 1951

Walter Wrigley

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INTRODUCTION

Inertial guidance of submarines can be considered as an extension of navigation techniques to self-contained systems that operate without reference to external information, save for initial set-up conditions. It is sometimes humorously referred to as "astronomy in a closet". In this respect inertial navigation can be looked on as a precision dead-reckoning method in which measurements are made with respect to inertial space* rather than with respect to the water surrounding the ship.

Conventional celestial navigation (see any text on navigation, e.g., Dutton(1)**) is a form of partial inertial navigation, in which stars represent the inertial space references. The requirement for submerged operation precludes the consideration of celestial systems in this report, however. In full inertial systems gyroscopes are used for inertial reference. The problem of navigation with self-contained systems has been surveyed by Draper and his collaborators(2), and the basic theoretical background of inertial guidance has been discussed by Wrigley(3).

The first part of this report treats the basic physical factors that are available for inertial navigation. This work is based primarily on that of reference (3), but particular emphasis is placed on aspects suitable for submarine operation. A discussion of the Schuler (84-minute) pendulum tuning condition, without which inertial systems would scarcely be practical, follows. A more complete treatment of the Schuler tuning condition has recently appeared(29). The final part of the report treats several basic systems for inertial indication of position. This treatment is idealized in that ideal components, e.g., "driftless" gyro units, are assumed, for ease in presenting the system fundamentals. This means that the only errors discussed in this report are those due to possible initial misalignment of the system. Accordingly, all systems treated are considered to be on an equal basis, and no attempt is made in this report to find a system most applicable to the submarine navigation problem.

In the next and final report these potential inertial systems will be examined from the practical point of view, i.e., from the point of view of the effects of actual instrumentation on system performance. In that report the effects of such characteristics as integrator drift and gyro drift will be considered, and an inertial system that appears most suitable for submarine navigation will be discussed in considerable detail.

It will be noted that the gyro units discussed in this report are of the single-degree-of-freedom type. Although it is fully recognized that the more conven-

* Inertial space is a space that can be associated with the "fixed stars". It is discussed in detail in this report.

** See list of references at end of text.

tional two-degree-of-freedom unit could just as well be used, it was necessary to restrict the discussion to one type, to prevent the report from becoming cumbersome, and because the former type of unit appears to be more widely used in present systems of this general nature.

The text contains only discussion and illustrations; all mathematical work is in the appended derivations. The notation used in the derivations is of the self-defining type, based on a formulation given by C. S. Draper, Notes on Instrument Engineering, M.I.T. Instrumentation Laboratory, Cambridge, Mass., September, 1950.

NATURAL PHENOMENA AVAILABLE FOR SUBMERGED NAVIGATION

Navigation by Radiation Data

Visual navigation, using recognized landmarks and associated maps, is a special class of navigation using radiation. It is necessarily subject to artificial and natural interference. In fact, the substitution of sound or lower frequency electromagnetic waves for light waves does not remove the interference limitation; all radiation systems have this in common. The interference-proof, self-contained navigation systems subsequently described, therefore, will be confined to the observation of natural phenomena associated with the Earth. The fundamental kinematics and dynamics of such systems are the subject of this report.

Utilization of Earth Phenomena

Navigation by Dead Reckoning

Three quantities available for submarine navigation by dead reckoning are the following:

1. static pressure,
2. dynamic pressure, and
3. temperature of the surrounding water.

They can be measured directly by a self-contained navigation system. From these data, the ship's depth and its velocity (in magnitude and direction) with respect to the water can be determined. However, the direction of the velocity vector with respect to a reference direction fixed in the Earth cannot be determined from these data. Such information requires, among other things, auxiliary directional indication, e.g., a gyrocompass. This permits the indication of position with respect to the water, although not with respect to the Earth. Position on the Earth is established when local ocean current data are used to modify the ship's velocity information. A complete self-contained system for navigation of a submarine anywhere on the Earth would then be achieved.

The reliability of these data will not involve merely the precision with which the system measures them. It is necessary also that position be uniquely determined. If the system is to indicate position anywhere on the Earth, there must be a one-to-one correspondence between any given set of data and a position on the Earth. More than this, Earth-fixed data which change in time—in particular, ocean current data — must be known at all times. This problem of uniqueness will, in general, be an important factor in the discussion of self-contained navigation systems.

The Earth's Electric Field as a Basis for Navigation

The Earth is the negative surface for an electrostatic field (4) extending outward into space. This field has an electrostatic potential gradient of about 130 volts per meter near the surface of the Earth and diminishes with altitude. Very little further information applicable to navigation is known. Thus, the Earth's electric field data are considered inadequate as the basis of a self-contained navigation system.

The Earth's Magnetic Field as a Basis for Navigation

The Earth's magnetic field has been used in the guidance of vehicles for centuries. This field has two poles, one in northern Canada and one in Antarctica, and effectively acts outside the mass of the Earth as though it were due to either a spherical magnet or an imbedded magnetic dipole.(5)(6) The field is normal to the surface of the Earth at the magnetic poles and tangent to the Earth at the magnetic equator. Accordingly, the magnetic field does not have a unique direction at any given point on the Earth. For example, at the magnetic equator the field is ideally everywhere parallel. Thus the magnetic field alone will not uniquely specify position on the Earth.

When auxiliary equipment is used to establish the horizontal plane, the magnetic field offers valuable means for establishing direction in the plane. This is done with the magnetic compass. Correction for the angular difference between magnetic north and geographic north, (declination or variation) utilizes the fact that the properties of the magnetic field are quite well known for all of the surface of the Earth, except in the polar regions.

The total intensity of the magnetic field is lowest at the magnetic equator and increases toward the magnetic poles, where it becomes approximately twice as large as at the equator. The total field intensity might therefore serve as an identification for position on the Earth.(7) However, such a location is not unique, because it furnishes only a locus of positions having a given intensity. (Such a locus is roughly a parallel of magnetic latitude). Auxiliary information is required to establish a unique position on the locus.

When an electric conductor is in motion relative to a magnetic field, a potential difference proportional to this velocity is induced between the ends of the conductor.(8) It has often been suggested that the current associated with such a potential difference could be used to indicate ground speed, i.e., a conductor could be moved so as to cut the "lines of force" of the Earth's magnetic field. The fallacy of the suggestion lies in the fact that the electromotive force required to drive such a current depends on the net rate of change of magnetic flux through a closed circuit.(8) In any circuit that can be carried in a vehicle, the Earth's magnetic flux is sufficiently homogeneous that there is no net change.* Therefore the expected current is always zero, and no ground speed information is available from this source.

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* It is to be noted that in a closed electric circuit moving in an homogeneous electromagnetic field, exactly as much flux enters the loop as leaves it in a given time.

Magnetic data are subject to interference by natural causes, notably magnetic storms. These storms appear to be closely associated with the aurora and sunspot activity.[9] While artificial interference with magnetic data cannot be ruled out, it does not appear likely at the present time because of the large amount of power required to generate any effective dynamic disturbance. A static disturbance would be encountered in the shielding effect of the hull of the ship carrying the navigation system. Accordingly, in a submerged submarine, the steel pressure hull would effectively shield out magnetic field data.

The Earth's Gravity Field as a Basis for Guidance

According to Newton's law of gravitation, every mass particle in the universe attracts every other particle with a force proportional to the product of their masses divided by the square of the distance between them. Any massive body, such as the Earth, can be considered as having associated with it a gravitational field. Such a field is essentially radial to the center of the Earth and has no poles on the Earth's surface. The possibility therefore exists for the direction of this field at any given point on the Earth to serve as a unique identification of the position of that point. The direction of the gravitational field could be readily determined by a plumb bob, if the Earth were non-rotating. The daily rotation of the Earth, however, produces a centrifugal acceleration that also affects the plumb bob. This is due to a basic law of physics, namely the principle of equivalence in the general theory of relativity, which states that gravitational mass is equivalent to inertial mass. Because of this, it is impossible to distinguish directly between gravitational effects and linear accelerations.[10] The vector resultant of the Earth's attracting gravitational field and the centrifugal force field due to the Earth's daily rotation is defined as the Earth's gravity field (see Fig. 1). The direction of this essentially radial gravity field is unique at any given point on the Earth. The secular (time) variation in the direction of the gravity field at a given point is caused mainly by tidal effects, and is less than 0.05 microradian.[11] For this reason the direction of the gravity field is a reliable, unique characteristic of any given point on the Earth. Furthermore, it is essentially impossible to interfere with gravity effects. Accordingly, the Earth's gravity field appears to be the most promising source of data for a self-contained navigation system.

The Earth itself is also subject to the equivalence of gravitational and inertial effects. If the Earth were a non-rotating fluid mass, it would form a sphere under the influence of its own gravitational attraction. Because of the Earth's structure and daily rotation, however, the associated centrifugal force field causes the Earth to bulge at the equator, giving rise to a nearly ellipsoidal or spheroidal shape. The figure that a fluid body with the mass distribution and daily rotation of the Earth would have is defined as the geoid, a "square shouldered" ellipsoid with a slightly undulated surface. The surface of the geoid is represented by mean sea level. Variations in

the elevation of the geoid relative to the closest reference ellipsoid are approximately one percent of the topographic variations in elevation. Since the geoid does not form an analytical surface, attempts are made to represent the geoid by an associated reference ellipsoid for the basis of geodetic measurements and map data. The Hayford Spheroid of 1909, which is the international reference ellipsoid,* has an equatorial radius of 6,378,388 meters and a polar semi-diameter of 6,356,912 meters, which gives an ellipticity** of 1/297.(11)(12)(13)

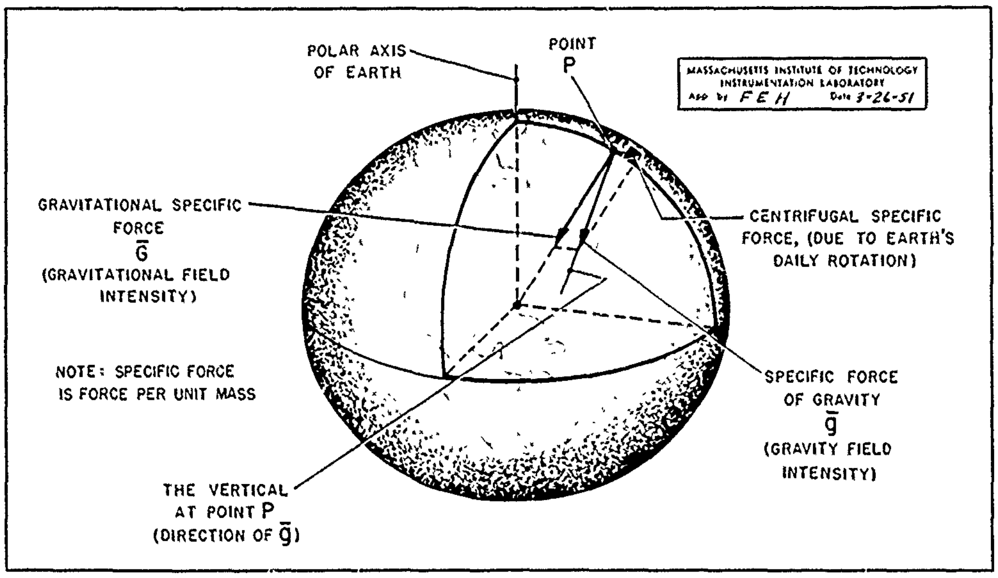

Figure 1. Gravitational Field Intensity and Gravity Field Intensity (the Vertical).

The surface of the geoid is everywhere an equipotential surface of the Earth's gravity field. The direction of the gradient of the gravity potential at the surface of the geoid is defined as the vertical. It is the direction of the specific force of gravity*** as indicated by a plumb bob with its base stationary with respect to the Earth (Fig. 1). The specific force of gravity (denoted by the vector $\bar{g}$) is the vector resultant of the gravitational specific force (denoted by the vector $\bar{G}$) and the centrifugal specific force associated with daily rotation.

Because the geoid does not have a consistently smooth surface, the vertical is generally not parallel to the normal to the reference ellipsoid

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* The Clarke Spheroid of 1866 is the basis for North American triangulations.

** Ellipticity is defined as the ratio of the difference between equatorial radius and the polar semi-diameter to the equatorial radius.

*** The specific force of gravity is defined as the force of gravity acting on a unit mass. It is identical with the gravity field intensity. Gravitational specific force is identical with the gravitational field intensity. Centrifugal specific force, which is a reaction force, is equal in magnitude but opposite in direction to the associated centripetal acceleration. Specific force, expressed in the dimensions of force per unit mass, is numerically and dimensionally equal to acceleration.

at the same position. The angle between the vertical and the ellipsoid normal is called the deflection of the vertical. It is generally less than 0.30 milliradian. In general, the largest marine deflections occur in the regions of rapid changes in ocean depth, such as near deeps or islands that rise abruptly from the ocean bottom.

The position of a point on the Earth's surface can be determined either by geodetic measurements, on which maps are based, or by astronomical position measurements, which utilize the vertical. (The measurement of astronomical position is subsequently discussed in this report.) Geodetic and astronomical position data will not agree, in general, because of the above-mentioned deflection of the vertical. However, position data determined by vertical indication and corresponding map data should agree in the case of small land masses such as islands whose position have been mapped from astronomical data.

In general, variations in the deflection of the vertical can be considered to be quasi-static at submarine speeds, and can be expected to have no dynamic effect on the operation of submarine inertial navigation systems.

GRAVITY FIELD NAVIGATION

Astronomical Position

The uniqueness of the vertical at any given point on the earth is the basis for the astronomical position[11] of the point. Astromonical position is an angle centered in the Earth and has two components, namely,

1. Astronomical latitude, the angle between the equatorial plane and the vertical at the located point. This quantity is unique because the Earth's polar axis has a unique spatial direction.

2. Astronomical longitude, the angle between the plane through the Earth's polar axis containing an arbitrary vertical (e.g., that at Greenwich) and the plane through the polar axis containing the located vertical. This quantity itself is not unique because of its arbitrary reference. However, difference of longitude is unique.

Inertial Navigation: Definition

Inertial navigation of a vehicle is defined as a method of locating the vehicle's position with respect to inertial space, using gravity field and acceleration data and Newton's laws of motion.* The concepts stated and implied in this definition will now be discussed.

Newton's first law of motion states that a body suffering no external forces is not accelerated, and therefore moves with constant velocity, i.e., constant speed in a straight line. The coordinate system to which this velocity is referred is called an inertial space. [10] [14] [15] Since, in particular, a centripetal force is not to act on the body, the coordinate system must be non-rotating. It can only move (with respect to some other coordinate system) along a straight line with constant velocity. Strictly speaking, it is impossible to distinguish in an absolute sense between two such coordinate systems, although Newtonian mechanics associates an absolute inertial space with the average positions of the so-called "fixed stars." However, special relativity theory associates the idea of an inertial space with any coordinate system moving at constant velocity relative to the material bodies of the universe. Any space fulfilling this requirement can be chosen as a reference inertial space. Because of the principle of equivalence in the general theory of relativity, it is necessary to include gravitational effects in determining an inertial space.[10][15]

Equivalence of Gravitational and Inertial Effects

One of the largest fundamental problems associated with inertial navigation is the indication of the vertical (e. g., the Earth's gravity field effect on a plumb bob; a gravitational** effect) inside a self-contained vehicle that

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* See any standard physical mechanics text for reference.

** For this illustration the difference between gravity and gravitation is ignored.

is suffering additional linear accelerations (inertial effects). The principle of equivalence in the theory of general relativity states that gravitational and inertial effects are intrinsically indistinguishable. A simple example [16] can illustrate this point: a man stands in an elevator, at rest with respect to the Earth, his feet on a platform scale. The scale reads his weight, i.e., the Earth pulls on him with a force equal to his weight; but he does not accelerate, because the scale pushes up on his feet with a force of the same magnitude as his weight. Now suppose the elevator accelerates upward, propelled by an external force. The scale now reads the inertia reaction force associated with this acceleration in addition to his weight. He observes that he weighs more; that is, it is as if the Earth's gravitational field had increased. If the external force is now removed and the elevator falls freely in the Earth's gravitational field, man, scale and elevator have the same downward acceleration. The scale reads zero, and it is as if the Earth's gravitational field were zero, as far as the man in the elevator is concerned. Thus, from within the elevator, an observer cannot distinguish gravitational from acceleration effects.

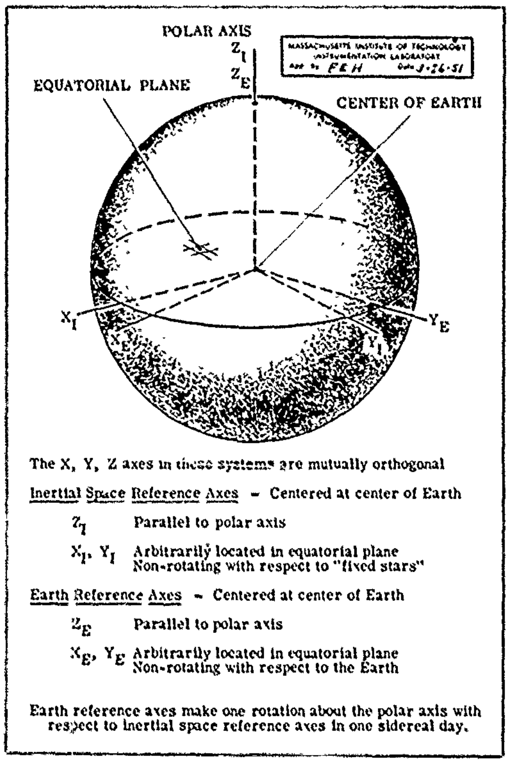

Figure 2. Geometrical Relationships between Earth Reference Axes and Inertial Space Reference Axes.

Coordinate Systems

For navigation purposes it would be desirable if acceleration could be measured directly with respect to the Earth. Unfortunately for the designer, the acceleration-sensitive equipment to be discussed makes its measurements with respect to inertial space, for which reason it is necessary to evaluate the several acceleration components that make up these measurements. This can be facilitated by a proper choice of coordinate systems, of which four adequately serve the needs of basic navigation:

1. Inertial space reference system
2. Earth reference system
3. Position reference system
4. Position system

Each of these is subsequently discussed in detail.

Inertial Space Reference System

The inertial space reference system must be one in which Newton's law of inertia is valid. This system can be most conveniently centered at the center of the Earth, which is allowable because the Earth is in an orbital, or free-fall, condition about the Sun*. The axes of this inertial space reference system must be non-rotating with respect to the "fixed stars". The Earth-centered inertial space reference system is illustrated in Fig. 2. Here, one of the three orthogonal axes, marked Z_I, is along the Earth's polar axis, and the other two, X_I and Y_I, are arbitrarily located in the plane of the Earth's equator, and are non-rotating with respect to the "fixed stars".

Earth Reference System

Earth reference axes are also most conveniently centered at the center of the Earth and are non-rotating with respect to the Earth. As shown in Fig. 2, one reference axis, Z_E, is selected to be coincident with the Earth's polar axis, and the other two axes, X_E and Y_E, are arbitrarily located in the plane of the equator. Earth reference axes make one rotation about the polar axis with respect to the inertial space reference axes in one sidereal day.

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* In an orbit, the central gravitational field exactly cancels the centrifugal field, thus leading to zero net acceleration.

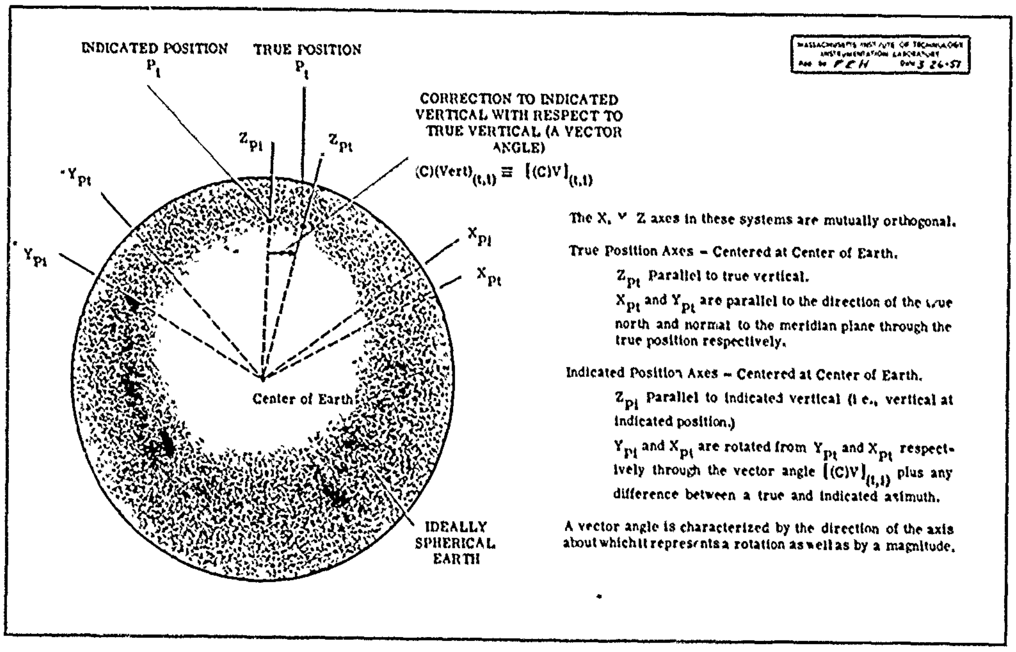


Figure 3. True Position and Indicated Position Coordinate Systems.

Position Reference System

The position reference coordinate system for navigation purposes will be taken as an Earth-centered system, the Z-axis coincident with the true vertical at the vehicle's present position. The Y-axis is arbitrarily taken as the normal to the meridian plane through the true position, while the X-axis is parallel to the direction of true north. These axes are shown as X_{Pt}, Y_{Pt}, and Z_{Pt} in Fig. 3.

Position System

The position coordinate system for navigation purposes will also be taken as an Earth-centered system, the Z-axis in this case being coincident with the indicated vertical at the vehicle's present position. Indicated axes (X_{Pi}, Y_{Pi}, Z_{Pi}, in Fig. 3) are identical with true axes when the indicated position and indicated azimuth reference* of the indicated position system are identical with the true position and true azimuth reference. When the two positions and horizontal references are not the same, the indicated axes are rotated from the true axes by the vector sum of two angles[17]:

1. The angle between the true vertical and the indicated vertical ($[(C)V]_{(t,i)}$ in Fig. 3)
2. The angle between the true horizontal reference and the indicated horizontal reference.

Practical Coordinate Systems

The coordinate systems illustrated in Figs. 2 and 3 serve well for the theoretical background because they are geocentric. In any actual navigation system, however, the coordinate origin will not be at the Earth's center, but in the vehicle, and the coordinate axes will be parallel to the earth reference axes or inertial axes. Moreover, additional coordinate systems will be required, some fixed in the vehicle, some in the indicating equipment, etc., to account for the actual conditions encountered in the solution of a navigation problem. Such conditions complicate the problems of the system designer, but they do not invalidate the conclusions reached in this paper for the idealized theoretical situation.

Theory of Vertical Indicators

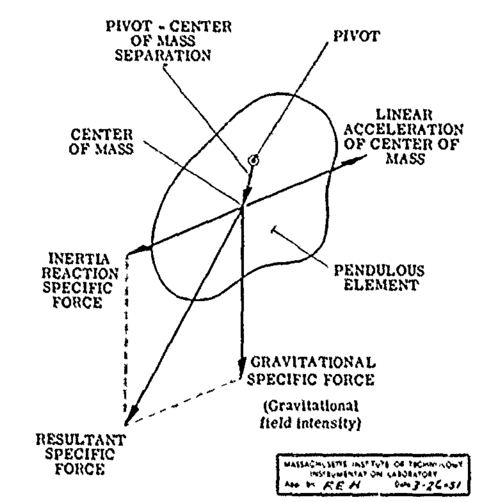
Specific Forces Associated with Gravity Field Navigation

The simplest vertical indicator is a plumb-bob or pendulum. In general, a pendulous element is defined as a system free to rotate about a point, so that it assumes a preferred orientation under the influence of impressed external forces. If the suspension-point is not accelerated with respect to the Earth, a pendulous element can provide a precise indication of the true

* Indicated azimuth represents a rotation about the indicated vertical.

vertical, i.e., the direction of gravity. If the suspension-point is accelerated with respect to the Earth, the pendulous element tends to indicate the apparent vertical, i.e., the direction of the resultant specific force.

When the pivot of a pendulous element is accelerated, a reaction force acts on the center of mass of the pendulous element. This force produces a torque that causes the pendulous element to "lag behind" the pivot. To study the effects of an acceleration without introducing the effect of the mass of the pendulous element, it is convenient to discuss the motion in terms of force per unit mass, i.e., in terms of the inertia reaction specific force associated with the acceleration. An inertia reaction specific force has the magnitude of its associated acceleration (although it will be expressed in force-per-mass units rather than in acceleration units) and is opposite in direction. Its direction is the one that a pendulous element seeks [18], and its projection on the input axis of a linear accelerometer is the direction in which the accelerometer element is urged. As mentioned earlier, the specific gravitational force is identical with the conventional gravitational field intensity. Figure 4 is a force diagram for an accelerated pendulous element in a gravitational field.



Pendulous element seeks to align its Pivot - Center of Mass Separation with the Resultant Specific Force.

Figure 4.
Specific Forces Acting on a Pendulous Element.

To obtain the specific forces acting on a pendulous element during a navigation problem it is necessary to know the associated accelerations. Page, in a reference[14], shows that the linear acceleration of a point with respect to a reference coordinate system may comprise five components* with respect to a coordinate system that is moving relative to the reference coordinate system**. These components are listed in column A at the left side of Table 1. The table lists also the results of Derivation 1 (appended) in columns B and C, giving the corresponding acceleration components of:

1. A vehicle whose true position coordinates move with respect to an Earth reference coordinate system; acceleration components are taken with respect to the Earth.

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* Wrigley has elsewhere [19] expanded this list to include simultaneous relationships between several coordinate systems.

** It is to be noted that the reference coordinate system may also be moving with respect to inertial space, but such motion is non-essential to this discussion.

TABLE 1

ACCELERATION COMPONENTS ASSOCIATED WITH SPECIFIC FORCES ON A PENDULOUS ELEMENT

A. A point in a coordinate system moving with respect to a reference coordinate system can have the following acceleration components:	B. A vehicle in a true position system moving with respect to an Earth reference system can have the following acceleration components with respect to the Earth:	C. A vehicle in a true position system moving with respect to an Earth reference system has the following acceleration components with respect to an inertial reference system:
1. Linear acceleration of one coordinate origin with respect to the other.	1. Zero; the two systems are concentric (Earth-centered).	1. Zero; the two systems are concentric (Earth-centered).
2. Linear acceleration of the point in the moving system.	2. Radial acceleration of the vehicle on the Earth, e.g., diving or surfacing of a submarine.	2. Acceleration with respect to the Earth, from Column B.
3. Centripetal acceleration when the moving system has an angular velocity with respect to the reference system.	3. Centripetal (radial) acceleration due to the angular velocity of the true vertical with respect to the Earth, caused by the linear surface velocity of the vehicle.	3. Centripetal acceleration parallel to the Earth-radius projection on the equatorial plane, due to the Earth's daily rotation; this, added to the gravitational field intensity, gives the gravity field intensity.
4. Tangential acceleration when the moving system has an angular acceleration with respect to the reference system.	4. Tangential (horizontal) acceleration due to the angular acceleration of the true vertical with respect to the Earth, caused by the linear surface acceleration of the vehicle.	4. Zero; the Earth rotates at an essentially constant angular velocity.
5. Coriolis acceleration when the point has a velocity in the moving system, which rotates with respect to the reference system.	5. Coriolis acceleration (forward, horizontal) due to rate of change of vehicle depth.	5. Three Coriolis acceleration components: a. transverse, horizontal; due to forward speed of the vehicle. b. eastward, horizontal; due to radial speed of the vehicle. c. vertical; due to Eastward speed of the vehicle.

2. A vehicle in a true position system stationary with respect to the Earth reference coordinate system; acceleration components are taken with respect to an inertial coordinate system.

As far as marine navigation is concerned, only two components in columns B and C of Table 1 are significant:

1. The tangential (horizontal) acceleration associated with the angular acceleration of the true vertical as the vehicle accelerates over the Earth's surface.
2. The centripetal acceleration parallel to the projection of the Earth-radius on the equatorial plane, due to the Earth's daily rotation; this, added to the gravitational field intensity, gives the gravity field intensity.

Geocentric Rotations

The discussion thus far has made repeated use of the concept of a rotating true vertical associated with the present position of the vehicle on the Earth's surface. This rotation of the vertical is geocentric,* and gives rise to two important acceleration components of the vehicle, as stated above. The concept is fundamental; essentially, all gravity field navigation systems are vertical indicators. The direct applicability of vertical indication to navigation stems from the already-mentioned fact that the vehicle's astronomical position can be expressed in terms of verticals taken at various points on the Earth. The application of a vertical indicator to navigation admits of what appear superficially to be two distinct points of view. In one case, the comparison of the indicated vertical with a reference vertical is used to get the desired latitude or longitude. The indicated vertical is then a datum involved in determining present position. According to the second point of view, the geocentric angular displacement of the vehicle is derived from double integration of the tangential acceleration measured by the navigation apparatus, and the vertical indicator plays the subsidiary but entirely necessary role of orienting the apparatus so that it will measure tangential accelerations only, i.e., it effectively provides a stable platform to eliminate the effect of gravity in the acceleration-measuring equipment. In either case, the basic inseparability of tangential and vertical accelerations requires the presence of the vertical indicator as such. These ideas will be discussed at length later.

Fundamentals of the Schuler (84-Minute) Pendulum

When a vehicle moves over the nearly spherical earth, the vertical associated with its instantaneous position rotates approximately geocentrically with respect to the Earth. On a ship moving at constant speed along a great circle over the Earth, a pendulum would have to rotate with respect to the Earth at an effectively constant angular velocity in order to indicate the vertical continuously. Once the transient

* Strictly speaking, this rotation is not truly geocentric, since the Earth is not actually spherical. However, in the subsequent text, this rotation will be referred to as "essentially geocentric".

stages were over, no torque would be required to keep the pendulum along the vertical during this rotation. On the other hand, if the ship were to accelerate, by changing either speed or heading, a torque would be required to maintain the pendulum on the vertical, which in turn is geocentrically accelerating.

Schuler [20] pointed out in 1923 that accurate indication of the vertical during periods of acceleration could be realized by suitable tuning of the natural period of the pendulum. Whenever the pivot of a pendulum is accelerated, the center of mass of the pendulum tends to "lag behind" the pivot with respect to inertial space. At the same time, this acceleration causes the true vertical, associated with the pivot, to accelerate geocentrically with respect to inertial space. These considerations lead to the following observation:

> If a pendulum initially hangs vertically, it will remain along the vertical <u>if its angular acceleration about its pivot equals the geocentric angular acceleration of the vertical</u>. (The accelerations are with respect to inertial space, as stated.)

These two accelerations become equal for a distributed-mass pendulum when the ratio of the pivot-center of mass separation to the square of the radius of gyration of the pendulum equals the reciprocal of the radius of the Earth. When operation is in the Earth's gravity field, and the pendulum is undamped, this condition gives a pendulum with a natural period of approximately eighty-four minutes; hence the term <u>84-minute pendulum</u>. For a shorter-period pendulum the rotation-producing torque on the pendulous element about its pivot is stronger, so the pendulum lags the true vertical. For a longer-period pendulum the torque is weaker and the pendulum leads the true vertical.

Derivation 2 presents a short mathematical treatment of the motion of a pendulum under the influence of a specific force. The dependent variable in the derivation is the correction that must be applied to the pendulum in order to align the indicated vertical, which is established by the pendulum, with the true vertical (see Fig. 2-1 for definitions and illustration of the quantities involved). This variable permits direct expression of the performance of a pendulum as an indicator of the vertical.

It is to be noted, from Fig. 2-1, that the horizontal inertia reaction specific force is equal to the negative of the product of the radius of the Earth and the angular acceleration of the true vertical. In other words, assuming that motion over the Earth involves operation over an essentially spherical surface, any linear acceleration of a point over the surface is inevitably accompanied by an essentially geocentric angular acceleration of the vertical representing that point. Because of the aforementioned principle of equivalence of gravitational and inertial masses, and assuming operation over a spherical surface, the orientation of the vertical (gravitational effect) and its angular acceleration (inertial effect) are inextricably bound together.

Clemens and his collaborators [21] have made an extensive study of the characteristics of the Earth-radius pendulum, i.e., the Schuler-tuned simple (concentrated mass) pendulum.

An 84-minute period is very long compared to the periods of physical pendulums ordinarily encountered. In fact, it is improbable that a simple (concentrated mass) pendulum or a distributed-mass pendulum could be constructed with this period, [20] although the 84-minute gyropendulum just fails of being realizable. However, in the practical case, a condition analogous to the Schuler tuning of a pendulum can be realized in a particular kind of closed-loop system in which a short-period pendulum tracks the specific forces. A platform, controlled on the basis of the pendulum output data, then becomes an equivalent Schuler pendulum. This will be discussed at length later.

It is to be noted that there is a definite functional relationship between the torque applied to a pendulum about horizontal axes and the linear acceleration of the unit. In a practical vertical-indicating system this torque-acceleration relationship may not hold due to added properties of the system, as is demonstrated in the subsequent discussion. Such a system is then not an actual Schuler pendulum but merely exhibits the Schuler tuning condition as one of its characteristics.

An implicit corollary of Schuler pendulum theory is that the direction with respect to inertial space of the resultant specific force remains fixed when the pivot of the pendulum undergoes a harmonic oscillation with an 84-minute period. In general, this means that the specific-force-tracking element in a Schuler-tuned inertial navigation system will be insensitive to an oscillatory disturbance having an eighty-four minute period.

The Pendulum as a Vertical Indicator: Summary

The following ideas are deduced from the foregoing:

1. A pendulum of arbitrary period tracks, i.e., tends to align itself with the apparent vertical. If the suspension-point of the pendulum is accelerated, the true vertical is then not available as a datum.

2. A pendulum with Schuler tuning effectively tracks the true vertical at all times.

3. The Schuler pendulum is <u>per se</u> physically unrealizable, chiefly due to a prohibitively small pivot-center of mass separation.

4. A vertical indicating system is conceivable which, while not a pendulum, utilizes the Schuler tuning idea, so that a torque caused by the ship's acceleration acting on the system will be absorbed by the inertia reaction torque required to keep a reference line in the system on the vertical; this is the fundamental objective of Schuler tuning.

Smoothing, Solution Time and Forced Dynamic Error

In general, no data received by an actual system are perfect, and the equipment will generate some uncertainty in its output. For this reason it is necessary to incorporate some degree of smoothing into any navigation system if the system is to be operated over extended periods of time. Smoothing, which has been extensively investigated in the fields of communications, automatic control, radar, and fire-control [22][23][24][25] involves the suppression of the roughness components of an input. Since roughness, or noise, generally appears among high-frequency Fourier components of a signal, smoothing means attenuation of the high-frequency end of the input spectrum, including legitimate high-frequency components of the input signal.

This high-frequency suppression "rounds off" the response of a system to sudden changes in the input, resulting in a delay in the system's achieving the state associated with new input conditions. One convenient measure of such a delay is solution time, defined for this report as the time for the system to achieve ninety-five percent of the change associated with a given input change [24]*. Solution time is important in determining the time required to remove transient effects associated with either the initial phase of a problem or sudden changes in the input quantity, i.e., the time required for the system to generate a "solution". In general, greater smoothing means a longer solution time.

When the input changes sufficiently rapidly with respect to system parameters, over an extended period of time, so that the output-input relationship is a function of the change, the delaying action of smoothing leads to a forced dynamic error. Increased smoothing increases the forced dynamic error.

The major problem facing the designer of an operating system is to determine the best compromise between the smoothing required to furnish an acceptable answer to the engineering problem, on the one hand, and on the other, the associated delay and forced dynamic error allowable in achieving the solution.

Pendulums and Accelerometers

Basically, pendulums and accelerometers both derive data from the resultant specific force acting on the system. The essential geometrical features associated with operation of a pendulum are illustrated in Fig. 5. A two-degree-of-freedom pendulum tracks the resultant specific force in such a manner as to try to align a tracking line, i.e., its pivot-center of mass separation vector, with the direction of the resultant specific force. The outputs of such a unit in a vertical indicating system are the angle

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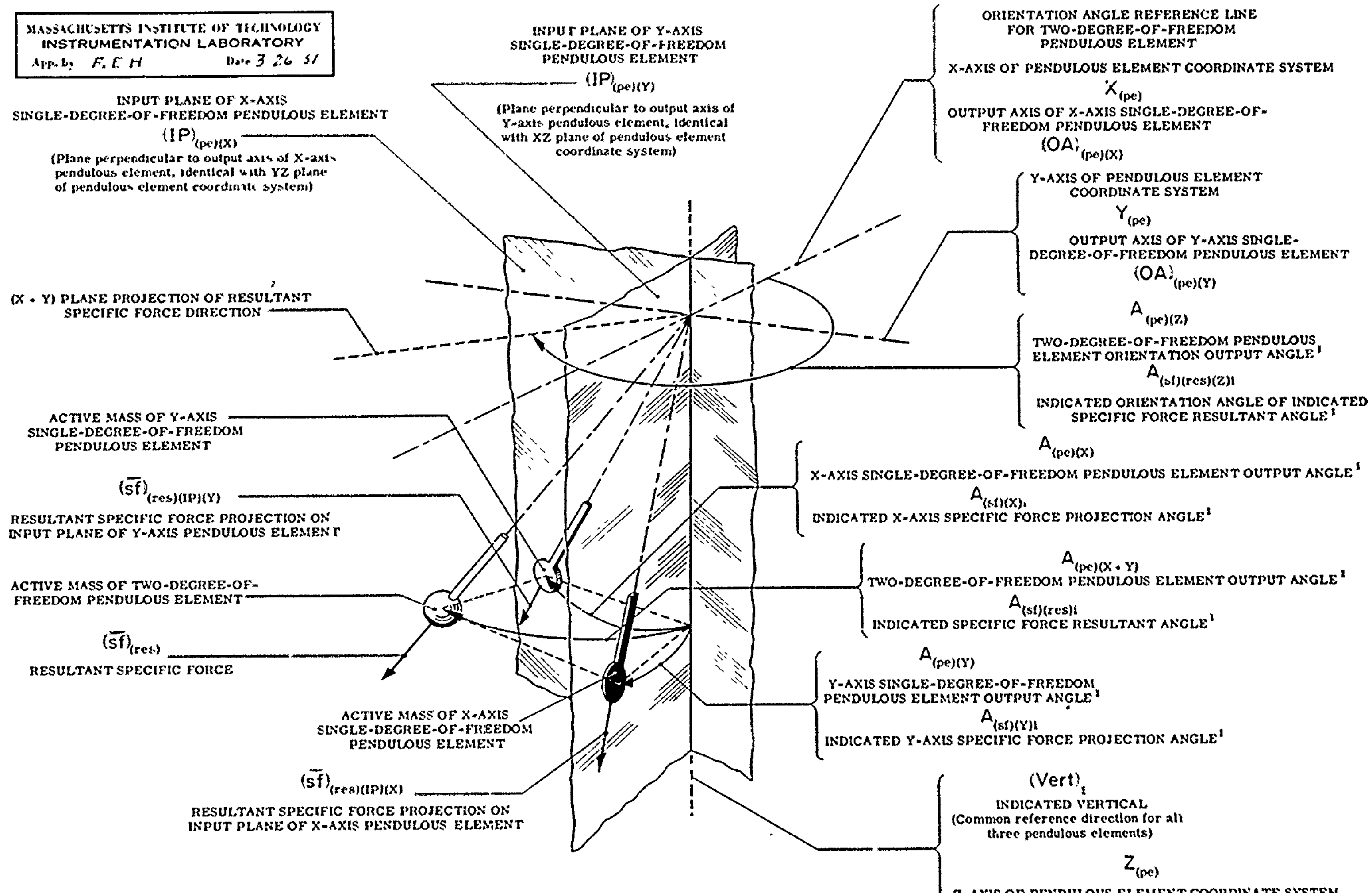
* This is equivalent to the time required to realize ninety-five percent of the area under the smoothing function[25].



[1] These angles are identical for the situation represented by this diagram, which is drawn for perfect specific force tracking of an ideal pendulous element.

Figure 5. Line Schematic Diagram Showing Performance Equivalence of Two Ideal Single-Degree-of-Freedom Pendulous Elements with Mutually Perpendicular Output Axes to One Ideal Two-Degree-of-Freedom Pendulous Element.

between the indicated vertical and the pendulum tracking line, and an orientation* of the tracking line as a rotation about the indicated vertical with respect to some reference in the indicated horizontal plane. A single-degree-of-freedom pendulum tracks the resultant specific force in such a manner as to try to align its tracking line with the direction of the projection of the resultant specific force on the plane normal to the axis of freedom of the pendulum. The output of such a unit is the angle between the indicated vertical and the pendulum tracking line about this axis. Two such units, with their axes of freedom nonparallel (preferably mutually perpendicular) in the indicated horizontal plane, are required for the complete indication of the vertical.

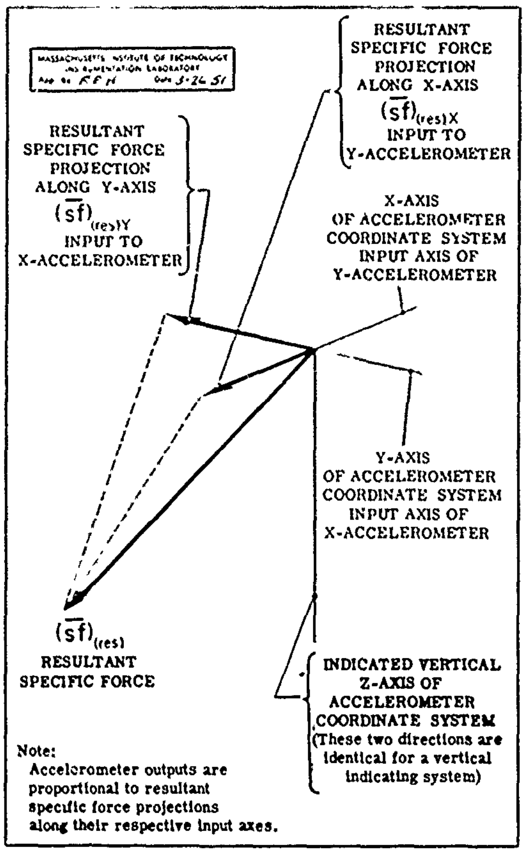


Figure 6. Specific Force Vector Diagram for Two Ideal Single-Degree-of-Freedom Accelerometers with Mutually Perpendicular Input Axes.

For comparison, a vector diagram associated with the operation of a linear accelerometer is given in Fig. 6. A single-degree-of-freedom accelerometer measures the component of the resultant specific force along its own input axis. The output of the unit is a signal proportional to the magnitude of this component. As with single-degree-of-freedom pendulums, two such units, with their input axes preferably mutually perpendicular in the indicated horizontal plane, are required for the full indication of the vertical. (Various methods of controlling the orientation of the platform on which the accelerometer or pendulum are mounted, on the basis of signals associated with the output of either device, are subsequently discussed.)

Note that the information obtainable from either pendulums or accelerometers is essentially the same. The nature of the information is

* The concept of an orientation as the mechanical input or output of a component deserves amplification here, since it will be applied repeatedly in this report. An orientation is defined as an angular displacement, and in the functional diagrams of this report includes all time derivatives of the displacement; i.e., instantaneous angular velocity, angular acceleration, etc.

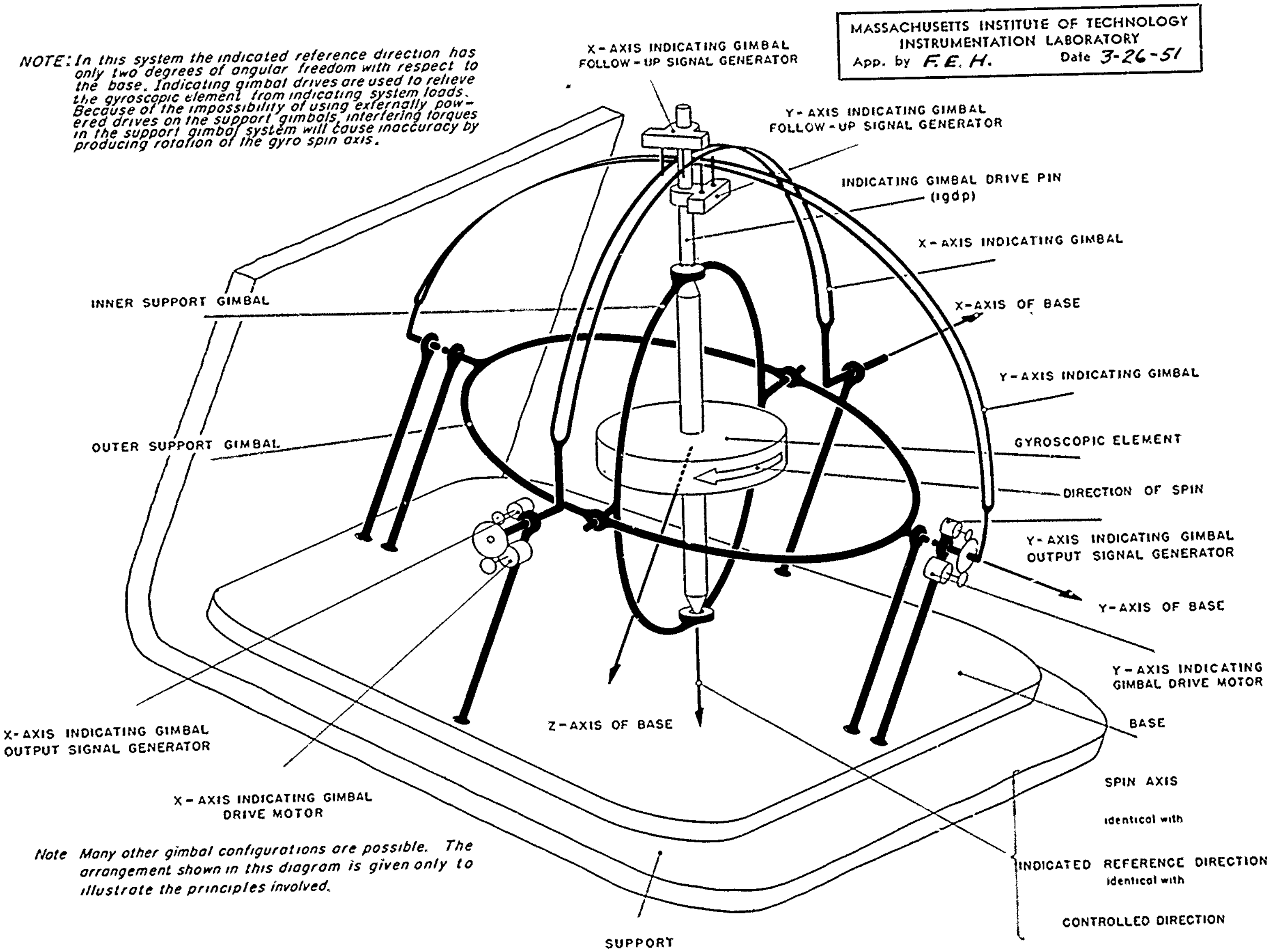


Figure 7. Line Schematic Diagram Showing Essential Features of a Single-Reference-Direction Stabilization Control Unit Using a Two-Degree-of-Freedom Gyroscopic Element.

different: a pendulum tracks a direction and an accelerometer measures a magnitude. Still, no basic information is obtainable from the use of one type of unit that cannot also be obtained from the use of the other. For this reason, it is not possible to use accelerometer data to correct pendulum data, or vice versa. The choice of unit is dictated by engineering design and operating requirements rather than by any basic difference in the data obtained.

It has been shown[3] that vertical-indicating performance is equivalent whether pendulums or accelerometers are used to obtain specific force data. With an accelerometer the Schuler tuning condition depends only on the magnitude of the radius of the Earth, and the system dynamics (natural period, damping, etc.) depend on the magnitude of the vertical component of the resultant specific force. With a pendulum, the Schuler tuning condition depends on the magnitudes of both the radius of the Earth and the vertical component of the resultant specific force, but the system dynamics are independent of either of these two factors. In all other performance characteristics the vertical-indicating systems are similar.

Gyroscopes: Basic Considerations

A detailed account of the performance of a gyroscope is given in a paper by J. J. Jarosh[26]. The essential geometrical features associated with gyroscope operation are illustrated in Figs. 7 and 8. A two-degree-of-freedom gyroscope (Fig. 7), to which no torques are deliberately applied and which has negligibly small uncertainty torques, will maintain the direction of the spin axis of its rotor fixed with respect to inertial space.* This is merely a manifestation of Newton's first law of motion applied to rotation. In this case, the angular momentum of the gyro directly furnishes the torque against which orientation stabilization can be made. The spin axis of such a gyro unit then becomes equivalent to a stellar line of sight, a property that is of prime importance to inertial navigation. When a torque is applied to a gyroscope, the gyro rotor unit rotates so as to align its spin axis with the applied torque at a <u>rate</u> proportional to the magnitude of the torque. This rotation is called <u>precession</u>, and is a manifestation of Newton's second law of motion applied to rotation. The gyro unit under such conditions serves as an integrator, yielding a rate of change of angular position as output with a torque as input. Follow-up signal generators are needed to furnish information on the orientation of the controlled platform that is using the two-degree-of-freedom gyro unit as its angular orientation reference.

A single-degree-of-freedom gyroscope must operate in conjunction with a closed-loop power-driven platform (Figure 8) that supports the gyro and is oriented on the basis of data received from the gyro. A careful distinction must be drawn between this gyro application and the one

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* Actually, it is the angular momentum that remains fixed in space. In a practical gyro unit, the mass distribution is such that the spin axis of the rotor is effectively a principal axis of inertia, and nutation is negligible; so that the angular momentum vector and the spin axis of rotor are effectively parallel.

just discussed, because here the gyro units themselves do not exert effective stabilizing torques; they merely act as signal generators. The performance of such a stabilization system depends on whether the gyroscope is of the integrating type or the rate type. An integrating gyro to which no torques are deliberately applied*, and which has negligibly small uncertainty torques, will maintain by means of its attendant closed-loop system an arbitrary direction fixed with respect to inertial space, in the plane normal to the input axis** of the gyro unit. Two such units, with their input axes preferably mutually perpendicular, can cause the direction of the normal to the plane of their input axes to remain fixed with respect to inertial space. This means that two single-degree-of-freedom integrating gyro units can cause a direction to remain fixed in inertial space in a manner equivalent to the maintenance of the direction by a single two-degree-of-freedom gyro unit.

A single-degree-of-freedom rate gyro operates in a manner similar to the analogous integrating gyro except that the maintenance of a direction in space depends on the nulling of the angular velocities acting on the platform rather than on the nulling of angular displacements. This mode of operation can lead to a drift of the direction that is being maintained away from the reference direction fixed in space. A platform controlled by single-degree-of-freedom gyro units (either rate or integrating; see Fig. 8) will rotate at a rate proportional to the torque applied to the gyro units, thus furnishing integrator or velocity-drive characteristics.

The method of using a gyro unit for space stabilization data has a marked effect on the theory of operation of the device. In one class of systems the gyros are left essentially undisturbed and maintain a reference direction in inertial space. In the other class of systems, torques are applied to the gyros to obtain a direction controlled by data from the system's tracking unit.

Methods for Indicating Astronomical Position

Basic Properties of a Feasible Vertical Indicator

In the preceding discussion of the Schuler pendulum, it was brought out that it is not physically feasible to use a pendulum directly as a precise indicator of the vertical under dynamic conditions. A practical system for indicating the vertical under dynamic conditions could, however, be made from a closed servo loop incorporating a form of Schuler tuning in the loop dynamics. The essential geometrical parts of such a loop are shown in Figure 9. This device will be called a vertical indicator. Its function is to maintain a platform called the controlled member in the horizontal plane. No further stabilization will be discussed here; specifically, the subject of the compass-like horizontal orientation of the platform will be deferred for later examination. There are three interrelated subsystems in the vertical indicator, two of which are

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* Torque supplied by rotation of the platform supporting the gyro unit are not classed here as deliberately applied torques; they are inherent results of the stabilizing process and cause null position operation of the gyro unit.

** The input axis of a single-degree-of-freedom gyro unit is the axis about which an angular velocity of the base supporting the gyro unit can cause a precession of the gyro spin axis about its single axis of freedom.

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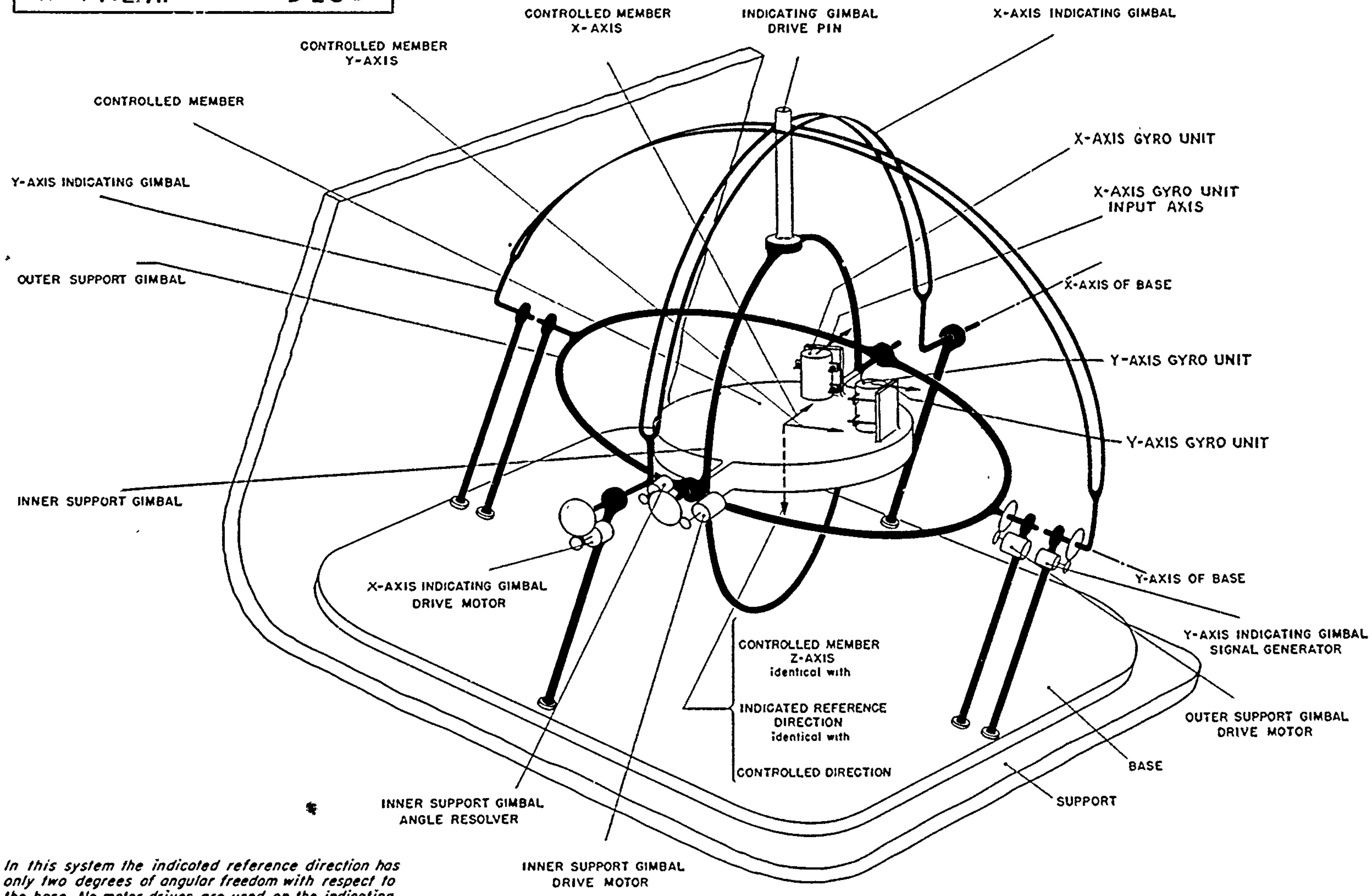
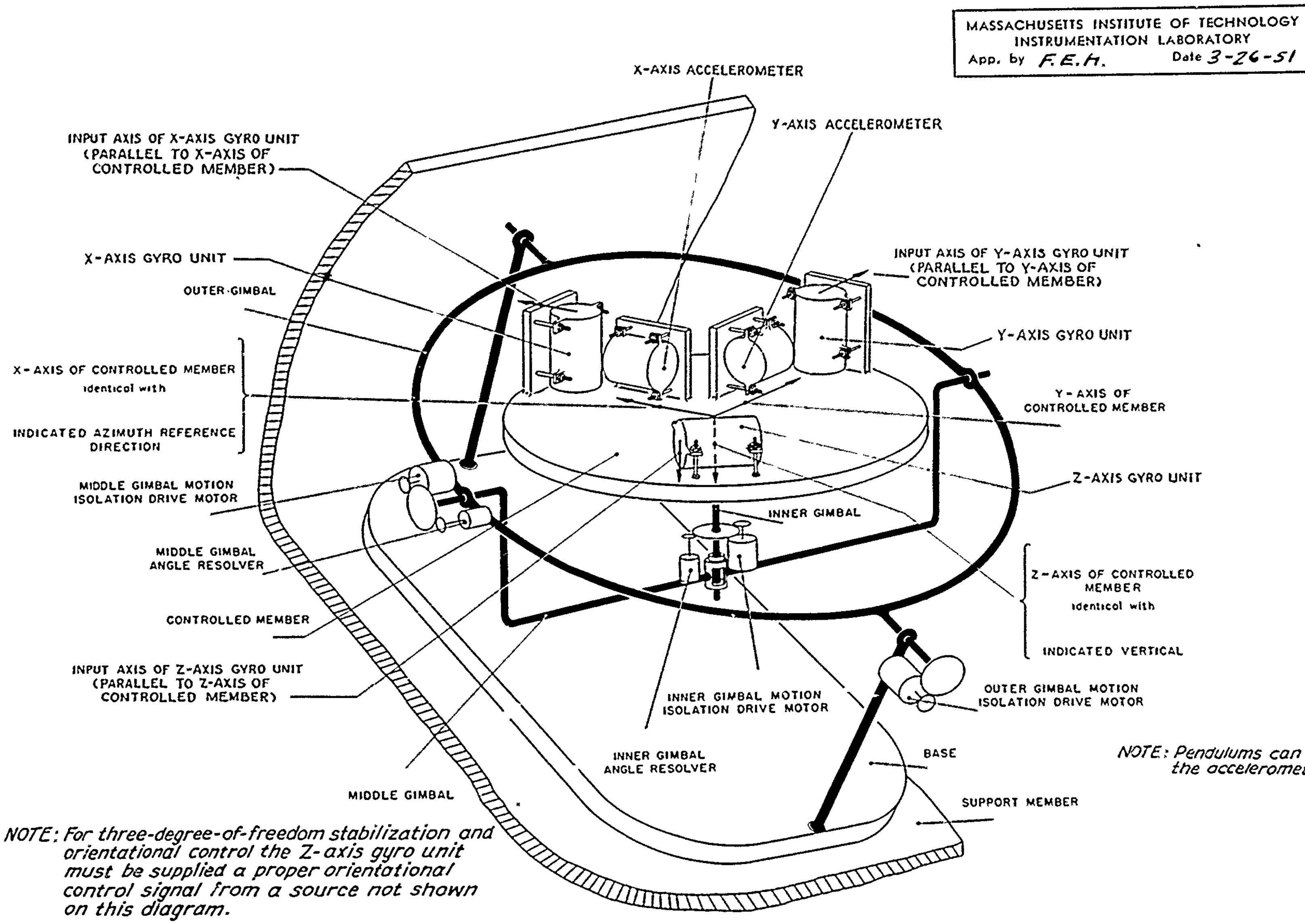


Note: In this system the indicated reference direction has only two degrees of angular freedom with respect to the base. No motor drives are used on the indicating gimbals. The support gimbal drives are sufficiently powerful to effectively eliminate interfering effects of torques required to position indicating and support gimbals.

Note - Many other gimbal configurations are possible. The arrangement shown in this diagram is given only to illustrate the principles involved.

Figure 8. Line Schematic Diagram Showing Essential Features of a Single-Reference-Direction Stabilization Control Unit Using Two Single-Degree-of-Freedom Gyro Units.



NOTE: Pendulums can replace the accelerometers.

NOTE: For three-degree-of-freedom stabilization and orientational control the Z-axis gyro unit must be supplied a proper orientational control signal from a source not shown on this diagram.

Figure 9. Line Schematic Diagram Showing Essential Geometrical Parts of a Complete Vertical Indicating System Using Two Single-Axis Accelerometers and Three Single-Degree-of-Freedom Gyro Units.

displayed in Figure 9. The three subsystems are as follows:

1. a system for obtaining resultant specific force data, involving either accelerometers or pendulums;

2. a system for orienting a controlled member in response to data from (1) to provide the vertical indication, involving gyro units* and servo drives; and

3. a system for properly modifying and filtering data from (1) above before they are applied to (2) in order to control the dynamic performance of the whole system.

The vertical indicating system shown schematically in Fig. 9 shows accelerometers and single-degree-of-freedom gyro units for subsystems (1) and (2). Subsystem (3) would consist of associated electronic and electromechanical equipment not shown in Fig. 9.

The Integration of Rates

When a system of the kind shown in Fig. 9 is carried over the Earth's surface, it follows a path that involves curvature with respect to space. In the case of the ideally spherical Earth, the path can be a great circle. A simple picture of the role rate integration plays in the indication of astronomical position is obtained by supposing the vehicle to have a meridian course (Fig. 10). The controlled member of Fig. 9 is started at the equator, the zero reference for latitude; as the controlled member accelerates northward along a meridian in the plane of Fig. 10, it suffers an essentially geocentric angular acceleration. Thus the controlled member itself must be re-oriented about its axes with respect to the Earth. For simplicity, suppose that its x-axis (Figure 9) is in the plane of Fig. 10; the controlled member then has an angular acceleration about its y-axis, which is perpendicular to the plane of Fig. 10. It receives this angular acceleration essentially in response to the sensing element mounted on it — an accelerometer or pendulum. Either device responds to the x-component of the inertia reaction specific force of the controlled member as well as being simultaneously responsive to the x-axis component of gravity.

A simplified vertical indicator which would serve this particular purpose is shown in Figure 11. The accelerometer necessarily detects linear accelerations, but the information from it must be processed to obtain an essentially geocentric angular displacement. This processing therefore involves a double integration with respect to time. The fact that the incremental angular displacement is essentially geocentric implies that a knowledge of the average Earth radius over the course would be required to furnish the distance of travel of the vehicle on the Earth, i. e., one minute of arc corresponds to one nautical mile.

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* In an actual system, the gyro units are part of the means for isolating the controlled member from arbitrary rotations of the supporting structure and vehicle, i. e., roll and pitch; this is taken up subsequently. Meanwhile, the role of the gyro units in the systems about to be discussed will be confined to their function as part of the controlled member drive system.

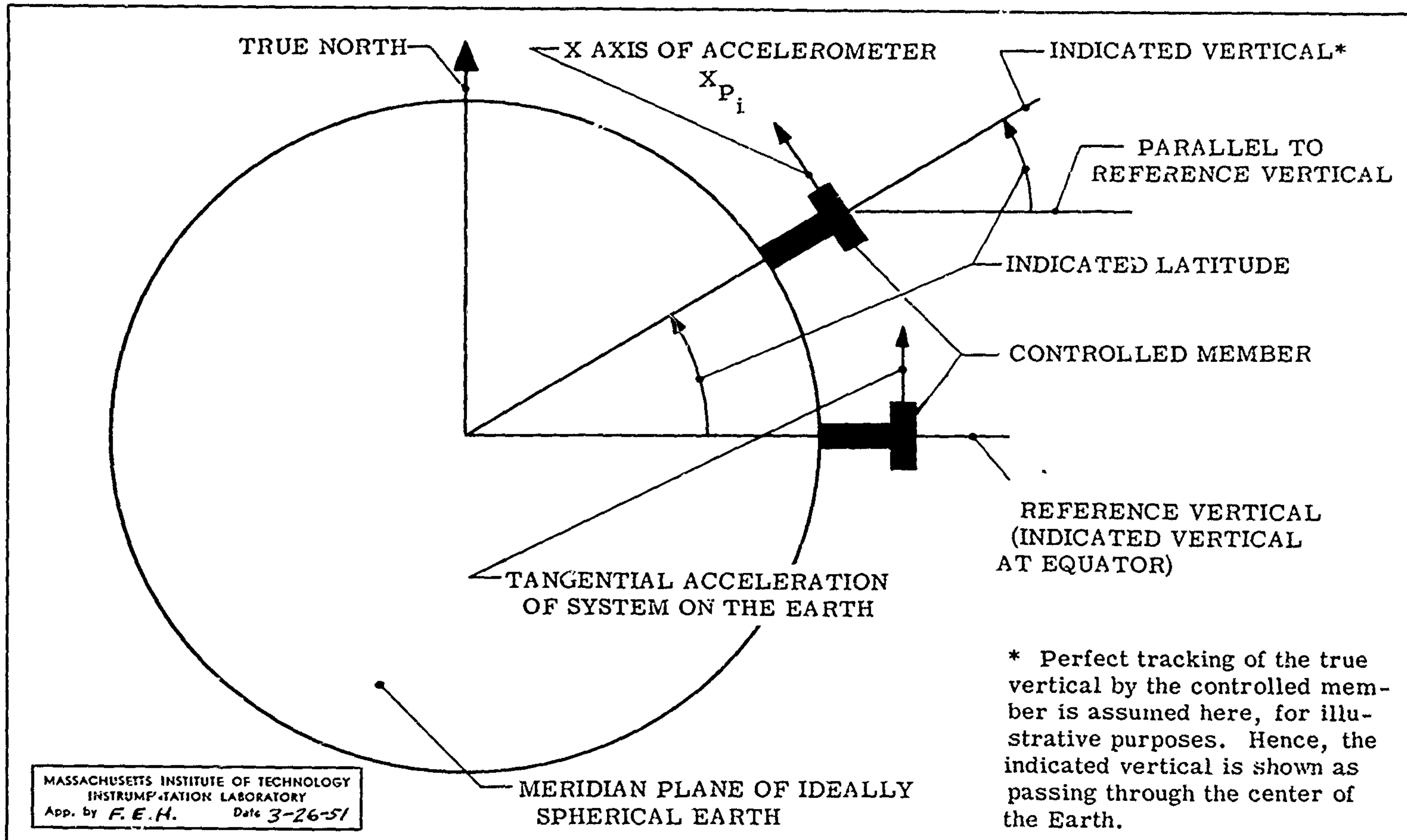


Figure 10. Vertical Indicator Accelerated Northward on a Meridian Course.

The integration of the accelerometer output data can be accomplished in two fundamentally different ways. These will be called analytical integration and geometric integration, respectively. Analytical integration is characterized by modifying the accelerometer output data by means of two integrators, so that the geocentric angular displacement is simply the output of the second integrator. The first integrator receives the angular acceleration of the indicated vertical as an input and delivers the angular velocity of the indicated vertical to the second integrator. As was mentioned previously in connection with the role of vertical indication in navigation systems, the accelerometer unit must be mounted on a stable platform, with its input axis tangential to the Earth, or it will indicate gravity components as well as the vehicle's linear acceleration on its course. This stabilization of the accelerometer mounting may then be regarded as a correction to the accelerometer input, since if the platform horizontal stabilization system is Schuler-tuned, it will tend to remove gravity effects from the accelerometer input.

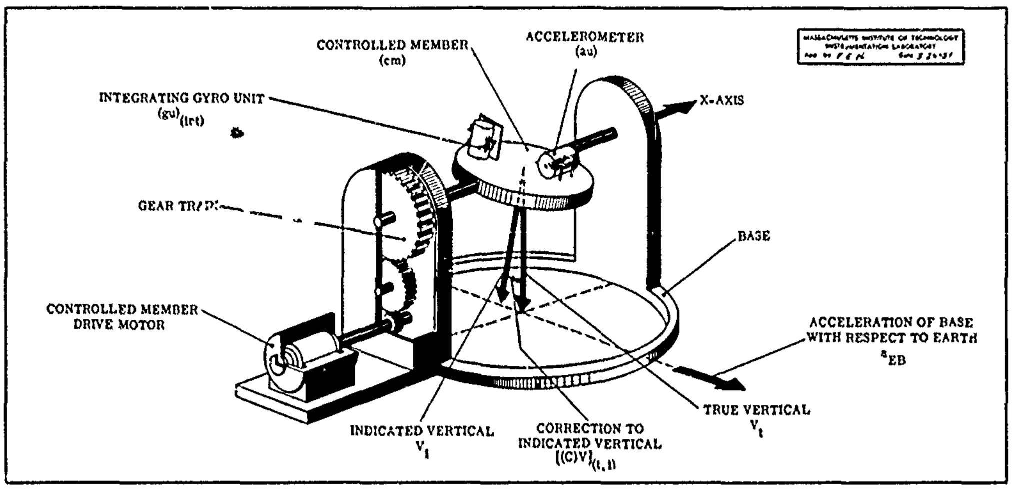


Figure 11. Line Schematic Diagram of Essential Operating Components of a Single-Degree-of-Freedom System for Indicating the Vertical When the Gravity Field Vector Lies in the Plane Normal to the X-Axis. The notation shown is used in Derivation 5.

Geometric integration, on the other hand, is characterized by comparing the indicated vertical with a reference vertical. In the simple case under examination, the reference vertical could be the direction of the indicated vertical at the equator. If this reference is physically carried with the system as it moves north, latitude will be indicated directly as the angle between the indicated vertical and the reference vertical. Evidently, the indicated vertical must be extracted as a datum from this system in order that the system indicate latitude.

Other essential differences between analytical and geometric integration of the accelerometer output data will be examined further in the discussion following.

The Basic Vertical Indicator

The method of integration for position information is immaterial as far as the orientation of the controlled member mounting the accelerometer is concerned: the controlled member must indicate the vertical*. Vertical indication is fundamental to the indication of geocentric angular displacement; the platform mounting the accelerometer must be tangential to the Earth. Before discussing various methods for determining the geocentric angular displacement, therefore, it will be of value to examine in some detail a simplified (i.e., ideal, undamped) single-degree-of-freedom vertical indicator of the kind sketched in Fig. 11. The extension of the following arguments to the two-degree-of-freedom vertical indicator (Fig. 9) then follows immediately. A functional diagram for the essential operating components of the single-degree-of-freedom system of Fig. 11 is shown in Fig. 12. This diagram is essentially duplicated as Fig. 3-2 of Derivation 3 appended. For the purpose of mathematical analysis there, a symbolic representation of the performance characteristics of the components is used to label them. In Derivation 3 the performance equations for this system are derived from basic physical considerations. The reader is referred there for a mathematical substantiation of the conclusions drawn in connection with the present discussion. It should be noted that only ideal, undamped systems are discussed in this report; non-ideal, damped systems are considered in the next report.

Schuler Tuning in a Closed-Loop System

The performance equation for the system diagrammed in Figure 12 is expressed in Derivation 3 in terms of the correction to the indicated vertical, which is the angle between the indicated vertical and the true vertical, i.e., this correction is the negative of the inaccuracy in indicating the vertical. The performance equation is a second-order differential equation which will contain a term in the angular acceleration of the true vertical unless the coefficient of this term is equal to zero. This term is part of the description of a system in which the indicated vertical consistently lags or leads the true vertical, and equating the coefficient to zero tends to remove the lead or lag, thus accomplishing an effective Schuler tuning of the system. Equating the coefficient of the true vertical's angular acceleration to zero amounts to adjusting the sensitivities of three components of the system so that their product is the reciprocal of the average Earth radius. This adjustment is applied to each of the systems to be discussed to obtain Schuler tuning.

In Derivation 3, an equation is derived which expresses the correction to the indicated vertical as a function of the time. As would be expected, considering the assumption that the system is undamped and Schuler-tuned, this performance equation is essentially that for a simple harmonic oscillator, with a period equal to the Schuler period, and with displacement and velocity amplitudes controlled by the initial conditions of the motion — i.e., the coefficient of the cosine term

* Except for certain short duration situations not applicable to general submarine navigation.

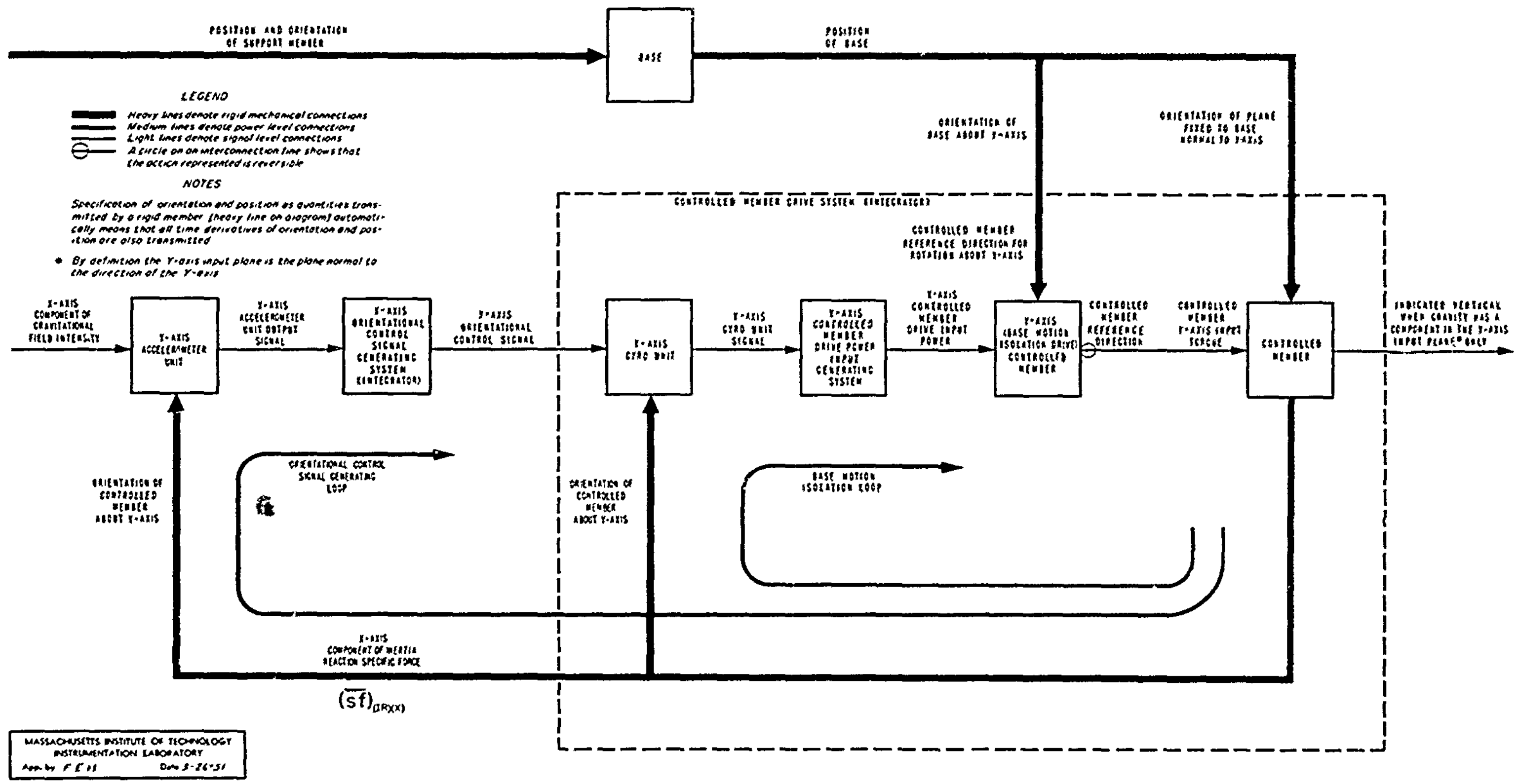


Figure 12. Functional Diagram Showing Interrelations Among Essential Operating Components of a Single-Degree-of-Freedom System for Indicating the Vertical when the Gravity Field Vector Lies in the Plane Normal to the Y-Axis.

is the correction to the indicated vertical at the start of the problem, and the coefficient of the sine term involves the time rate of change of this correction at the start of the problem.

It should be noted here that a superficially different point of view is possible concerning the operation of a vertical indicator: it may be said to "track" the indicated vertical. Resultant acceleration is the basic input to an accelerometer or a pendulum. Now when a pendulum support is accelerated, the pendulum bob tends to lag behind the support. In the most general case, this "lag" of the pendulum bob results in indication of the apparent vertical instead of the true vertical, where the apparent vertical is the direction of the resultant specific force and lags the true vertical by an angle that is a function of the acceleration acting. Consider that the vertical indicating system is a tracking device that receives resultant specific force as its input and gives the indicated vertical as its output. The input receiving member is a damped pendulum, which is a very satisfactory "tracker" of the apparent vertical. The controlled member is the tracking member; the direction of a normal to the plane established by the pendulum unit input axis is defined as the indicated vertical. If the indicated vertical is to be parallel to the true vertical, it is essential that the indicated vertical be displaced from the apparent vertical by some angle. The same considerations apply if an accelerometer replaces the pendulum mentioned above. From this point of view, the vertical indicator is a closed-loop system with a "lag" between output and input, which is a fundamental and necessary characteristic of its operation.

Fundamentals of Position Indication: Summary

Astronomical position measurement involves the determination of the essentially geocentric angles between indicated and reference verticals. It is proposed to do this with a stabilized platform as an essential feature. The stabilization loop receives specific force data from accelerometers or pendulums, and delivers these data to gyro units and servo drives, which in turn drive the platform toward a horizontal position. The specific force data are modified and filtered to control the system dynamics. Part of this modification consists of Schuler-tuning the stabilization loop. Two basic methods of utilizing the indicated vertical are then available to obtain the vehicle's essentially geocentric angle of travel. One, called here analytical integration, involves the integration of accelerometer outputs or of gyro inputs, with the accelerometers and gyro units mounted on the stable platform. The other method, called here geometric integration, involves the direct comparison of the indicated vertical with a reference vertical. Both methods will now be outlined in terms of six representative systems.

Typical Methods for Inertial Indication of Position

Open-Chain Integration of the Angular Velocity of the Indicated Vertical

Figure 13 is a functional diagram of the essential operating components of a latitude indicator which uses an open-chain single integration of the angular velocity of the indicated vertical to arrive at the indicated latitude.

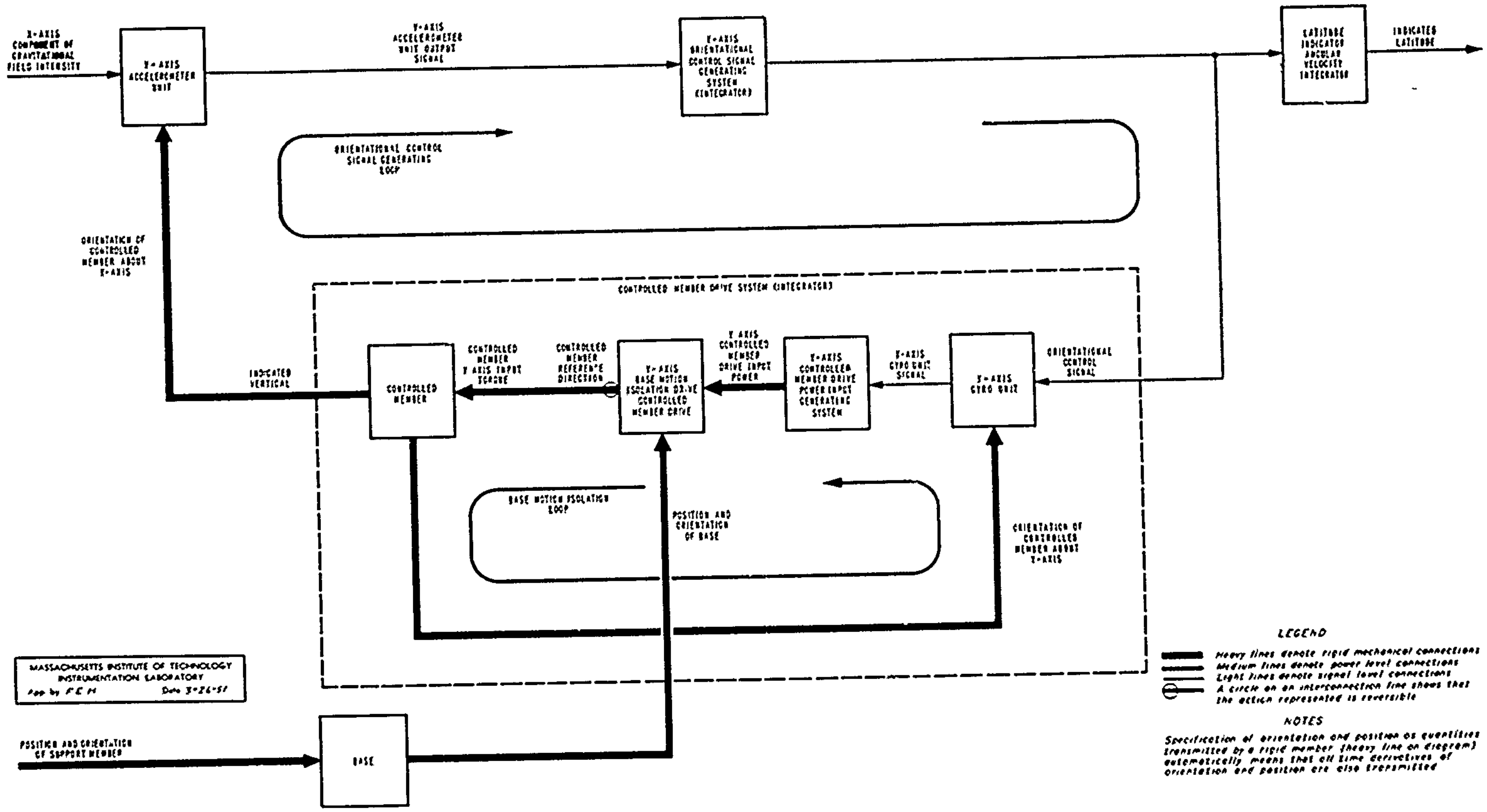


Figure 13. Functional Diagram Showing Interrelations Among Essential Operating Components of a Latitude Indicator, Using an Open-Chain Single Integration of the Angular Velocity of the Indicated Vertical.

Figure 13 corresponds to Fig. 4-1 in Derivation 4. Note that this system is primarily the basic vertical indicator of Fig. 12, with the angular velocity integration tapped off the vertical indicator loop at the angular velocity signal that calls for precessing the gyro unit, so that the position integrator output is the indicated change in latitude. The vertical indicator is undisturbed by the presence of this integrator, as far as the loop dynamics are concerned. Hence, (as is shown in Derivation 4) the same Schuler tuning condition applies to this system as applied to the basic vertical indicator of Fig. 12.

A two-degree-of-freedom system constructed along these lines would require an x-axis accelerometer in addition to the y-axis accelerometer shown in Fig. 13, as well as x-axis components corresponding to the orientational control signal generating system, gyro unit, and controlled member drive. In addition, the x-axis system requires that its angular velocity signal be modified by secant of latitude in order to give longitude rate data to its position integrator. The position integrator in the x-axis system would indicate longitude difference with respect to some arbitrary reference. The maintenance of the y and x axes along a parallel of latitude and a meridian, respectively, must be handled by an auxiliary orientation system, e.g., a gyrocompass.

The gyro units ideally maintain their spin axes fixed with respect to inertial space. To maintain the controlled member on which they are mounted tangential to the Earth in its daily rotation, therefore, the x-axis gyro must be precessed by a correction torque which keeps the platform up with Earth rate. The calculations of Derivation 4 consider this torque to be generated by a special precession current applied to the torque generator of the gyro unit.

In Derivation 4, it is shown that if the controlled member has the equation of motion of a Schuler-tuned pendulum, the correction to the indicated position (that is, the instantaneous difference between the true and indicated positions) has a similar equation of motion, except for the appearance of certain initial conditions as additive constants of integration in the indicated position. Specifically, the correction to the indicated position at any time is the correction to the indicated position at the start of the problem, plus the present correction to the indicated vertical, less the initial correction to the indicated vertical. The indicated position is thus Schuler-periodic with respect to boundary-matching terms associated with the vertical indication.

Open-Chain Double Integration of the Angular Acceleration of the Indicated Vertical

Figure 14 is a functional diagram of the essential operating components of a latitude indicator which uses an open-chain double integration of the angular acceleration of the indicated vertical to arrive at the indicated latitude. Figure 14 corresponds to Fig. 5-1 of Derivation 5. This system represents only a slight modification of that of Fig. 13, and the prior discussion con-

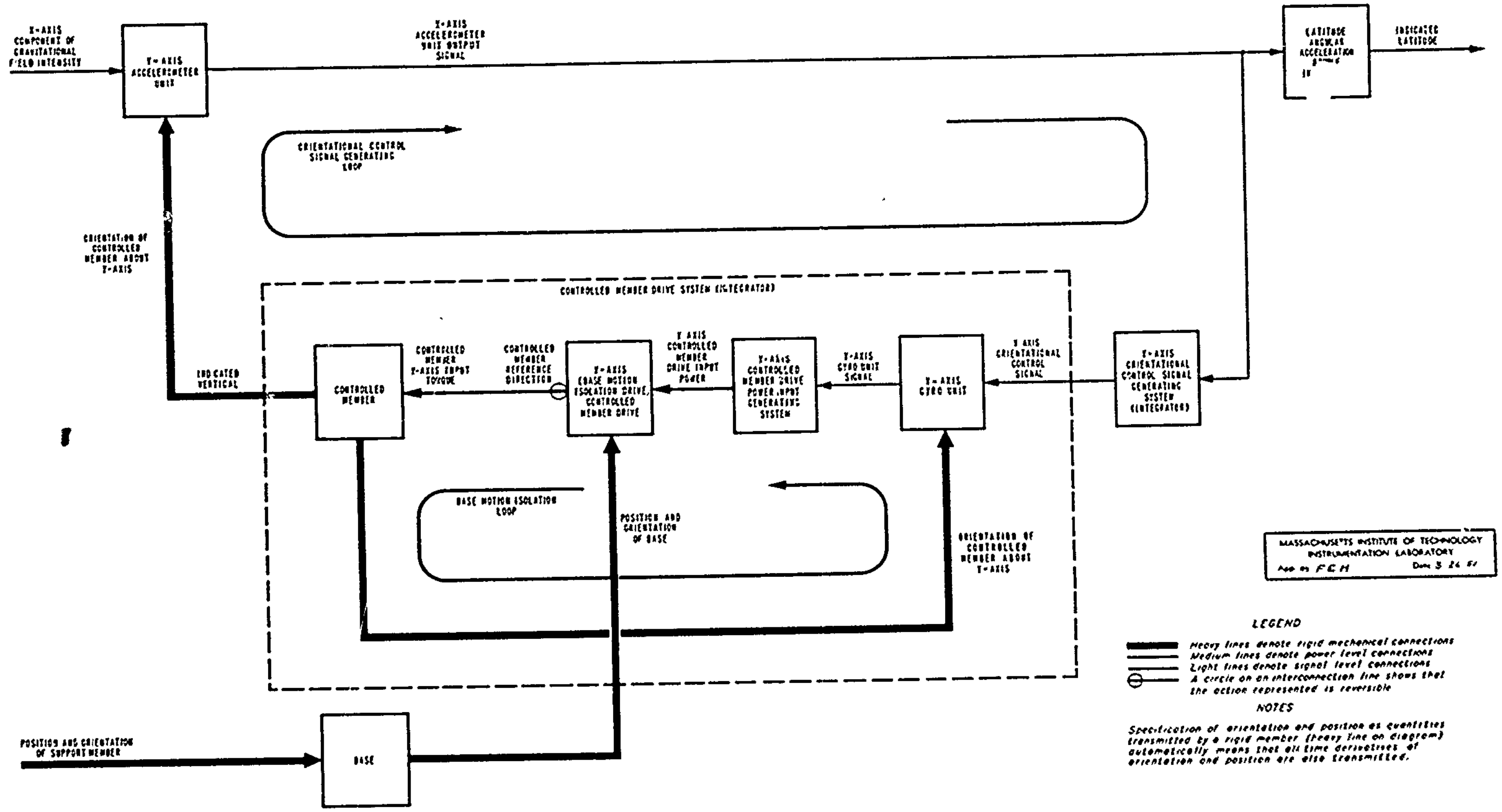


Figure 14. Functional Diagram Showing Interrelations Among Essential Operating Components of a Latitude Indicator, Using an Open-Chain Double Integration of the Angular Acceleration of the Indicated Vertical.

cerning Schuler tuning, the extension two degrees of freedom and longitude as well as latitude indication, and the gyrocompass requirement, is applicable. It should be noted that while the accelerometer-unit itself indicates linear acceleration, the desired system output is an essentially geocentric angle of travel. The accelerometer output is, in fact, proportional to an angular acceleration, namely, the linear acceleration of the vehicle divided by the average Earth radius over the course. This Earth-radius value is important in the Schuler tuning of the system, as described in Derivation 3, and determines the dynamics of the vertical indicator. However, it is emphasized that the Earth radius has no other role in the indication of position. In effect, by stabilizing the platform, the Schuler tuning condition furnishes the average Earth radius at which the accelerometer unit operates, so that an essentially geocentric angular acceleration is immediately derivable from its output.

The performance equation for this system is given in Derivation 5. The correction to the indicated position involves:

(a) the initial correction to the indicated position
(b) the initial time derivative of the correction to the indicated position; this term will cause the correction to increase directly with the time of operation, since it represents a false ground speed component
(c) the initial correction to the indicated vertical; the correction is Schuler-periodic with respect to this term
(d) the initial time derivative of the correction to the indicated vertical; this term appears as part of two functions: the first will produce Schuler-periodic oscillations, and the second will cause the correction to increase directly with time of operation.

Direct Double Integration of Acceleration

Figure 15 is a functional diagram of the essential operating components of a latitude indicator which uses a doubly-integrating accelerometer as an inherent component in the vertical-indicating loop. The output of such an accelerometer is immediately proportional to the change in indicated latitude. However, this output is not a suitable orientational control signal for the gyro unit, which must be driven by a signal corresponding to the angular velocity of the indicated vertical. The vertical indicating loop for this system, as distinguished from the three systems previously taken up, therefore contains a differentiator between the accelerometer and the gyro unit. This slightly modifies the expression for the Schuler tuning condition for this loop, in that the differentiator sensitivity appears instead of the integrator sensitivity used in the loops of Figs. 12, 13, and 14. Otherwise, the Schuler tuning condition is unchanged. The extension of this description of a single-axis system to that involving two degrees of freedom goes as in the case of the system of Fig. 14. A gyrocompass or similar azimuth device is also a requirement here. Derivation 6 shows that the equation of motion of this position indicator is the same as for the system utilizing Open-Chain Double Integration. The correction to the indicated position will again involve the same four quantities resulting in terms with Schuler-periodic oscillations plus terms that build up with time.

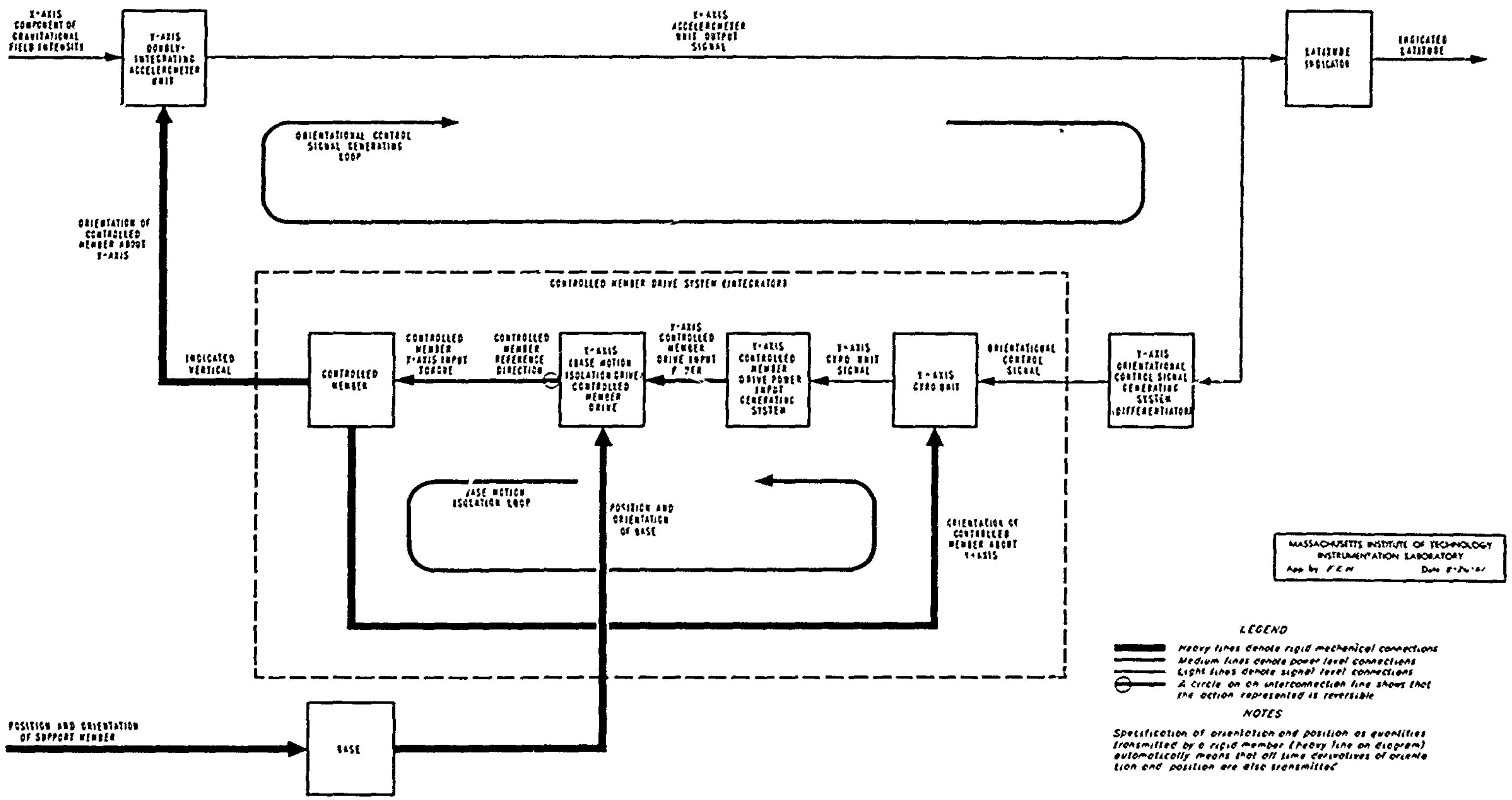


Figure 15. Functional Diagram Showing Interrelations Among Essential Operating Components of a Latitude Indicator, Using a Doubly-Integrating Accelerometer.

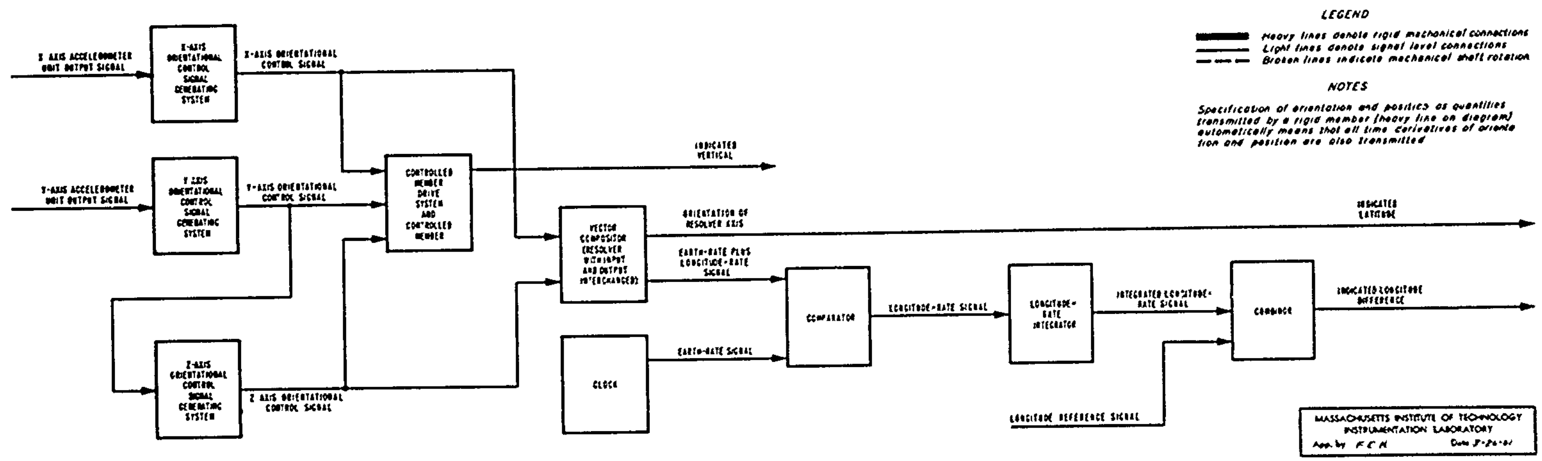


Figure 16. Partial Functional Schematic of a Three-Axis Position Indicator Using the Components of Figure 9. The X and Y systems are similar to Figure 10. Two components of celestial longitude rate are composed into the corresponding vector and vector angle.

Electromagnetic Composition of Tangential and Vertical Components of Celestial Longitude Rate

If the system of Fig. 9 is stabilized in azimuth, by means to be taken up subsequently, two components of celestial longitude rate (longitude rate taken with respect to inertial space) may be used to obtain indicated latitude and indicated longitude difference. Figure 16 is a partial functional diagram of such a system. The signals feeding the x-gyro unit (Fig. 12) and the z-gyro unit (Fig. 9) are respectively proportional to these celestial rate components; the x-signal is proportional to a quantity given by the sum of Earth-rate and longitude rate multiplied by the cosine of indicated latitude, and the z-signal is proportional to the same quantity multiplied by the sine of the indicated latitude. These statements are justified in Derivation 8, and the extraction of position information from these data is also covered there in detail. In brief, the method is as follows: the aforementioned cosine and sine component signals are fed to a trigonometric resolver (see Fig. 8-1 in Derivation 8). Under these input conditions the resolver acts as a device which composes the sine and cosine component inputs into the magnitude of the vector (Earth-rate plus longitude rate), and the resolver rotor is automatically rotated by an associated servo to the angle corresponding to the sine and cosine terms; indicated latitude, in this case. The magnitude of the vector must be integrated to obtain longitude information. Furthermore, the Earth-rate term must be removed, either before or after the integration, to make this information represent longitude difference. The integrated longitude rate, i.e., longitude difference, is then referred to a reference longitude, so that the second output of the system is indicated longitude. The mechanization of an ideal system performing these functions is detailed in Fig. 8-2 of Derivation 8.

Geometric Integration Using Angular Velocity of the Indicated Vertical with Respect to Inertial Space

Another approach to the position-indicating problem, using geometric integration, is shown in the form of a functional schematic diagram in Fig. 17, corresponding to Fig. 9-1 of Derivation 9. The method is as follows: A stable platform is set up, indicating the vertical — essentially the two-axis version of Fig. 9. This platform is then oriented by a gyrocompass, by a method to be described, establishing a geographic north line. On this stable platform, a latitude gimbal is erected. This gimbal has an axis of rotation in the horizontal east-west direction (Fig. 18) and is automatically elevated about this axis on the basis of angular velocity information in the indicated meridian plane. This orientation is accomplished by the latitude indication loop (Fig. 17). In equilibrium, the latitude gyro unit, which is the angular velocity tracker, has its input axis perpendicular to the indicated Earth's polar axis. When the latitude gimbal, on which the gyro unit is mounted, assumes an orientation other than this equilibrium orientation, the gyro unit (a rate gyro) receives an angular velocity component input. The gyro unit then generates a signal which, after integration, is used to rotate the latitude gimbal to its equilibrium orientation, thereby causing null operation of the rate gyro. Latitude is then directly

indicated (Fig. 18) as the angle between the latitude gimbal and the indicated vertical. The performance equation for this system is developed in Derivation 9.

Within the latitude gimbal, the longitude gimbal (Fig. 17) rotates about the indicated Earth's polar axis to register indicated celestial longitude difference, or celestial longitude with respect to some arbitrary sidereal reference. The longitude gimbal supports the longitude gyro unit (an integrating gyro). The gyro unit, which has its input axis parallel to the indicated Earth's polar axis, receives indicated celestial longitude rate as an input. The gyro unit then generates a signal which is used to rotate the longitude gimbal relative to the latitude gimbal at the negative of the indicated celestial longitude rate. In other words, the longitude gimbal tends to remain fixed in inertial space. In the position-indicating system just discussed, an electromagnetic vector compositor yielded celestial longitude rate as an output, which was then processed to obtain indicated longitude difference, by the subtraction of Earth-rate and subsequent integration. In the present case, Earth-rate compensation is also required to obtain longitude difference*, but the required integration is inherent in the operation of the longitude drive system. The performance equation for this system is also developed in Derivation 9. It is shown there that the correction to the indicated latitude consists of sinusoidal terms involving a period dependent on the angular momentum of the latitude gyro, certain system sensitivities, celestial longitude rate, and a term involving the acceleration in latitude.

It is to be noted that this entire system is self-settling; the only initial setting required is the longitude reference, which can be conveniently made at the longitude indicating system.

Geometric Integration Using a Pre-Aligned Inertial Reference

Figure 19 is a functional diagram of the essential operating components of a latitude indicator which compares the indicated vertical with a pre-aligned reference vertical parallel to the Earth's polar axis. The reference vertical is maintained by a pre-aligned rate-integrating gyro unit and its associated servo drive. The gyro unit is left undisturbed during the marine operations, thereby maintaining an inertial space reference orientation. Figure 19 corresponds to Fig. 10-1 in Derivation 10. The superposition of the outputs of two indicators — one vertical indicator of the tracking type shown in Fig. 12 and one fixed-axis reference indicator maintained by an inertial gyro — yields indicated co-latitude. The fixed-axis indicator furnishes essentially the indicated polar axis.

---

* Two practical general methods of compensation are:

1. A sidereal time drive (clock), used to rotate the longitude gimbal about the indicated polar axis, which otherwise indicates celestial longitude difference, to remove integrated Earth-rate and leave longitude difference.

2. An angular data transmitter on the longitude gimbal, used to remove celestial longitude data to a remote point, where the orientation of the data receiver and the orientation of a sidereal clock shaft feed a differential whose output is indicated longitude difference.

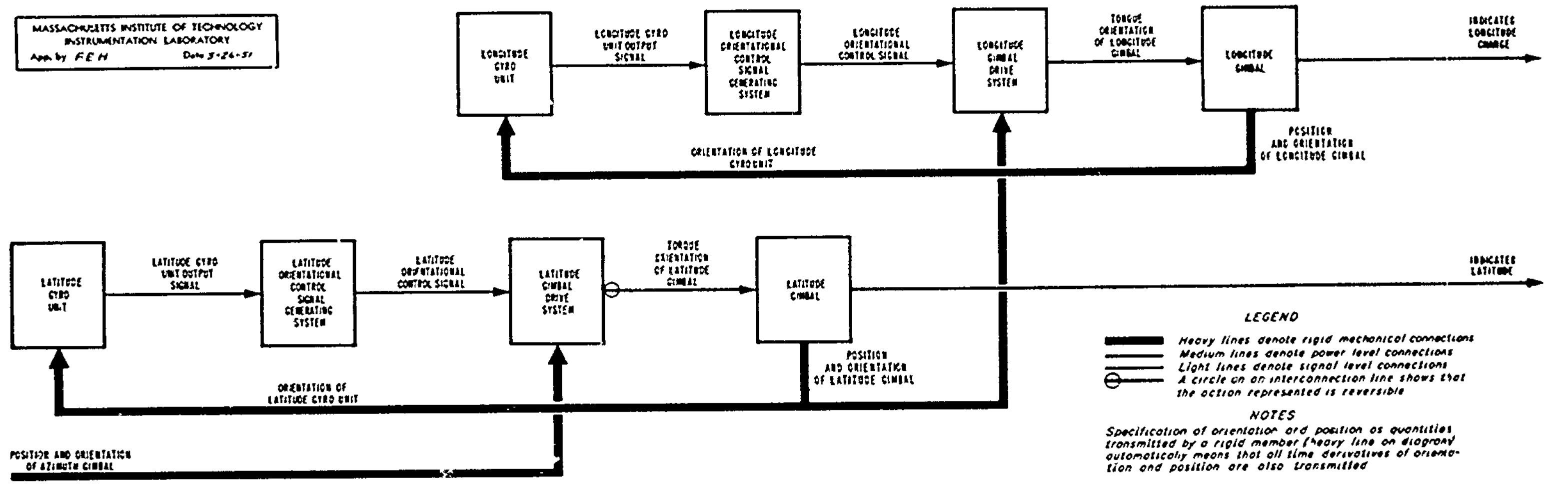


Figure 17. Partial Functional Schematic Diagram of a Three-Axis System for Position Indication Using Detection of Tangential and Horizontal Earth-Rate Components, Using the System of Figure 5 to Furnish Three-Axis Stabilization in Azimuth.

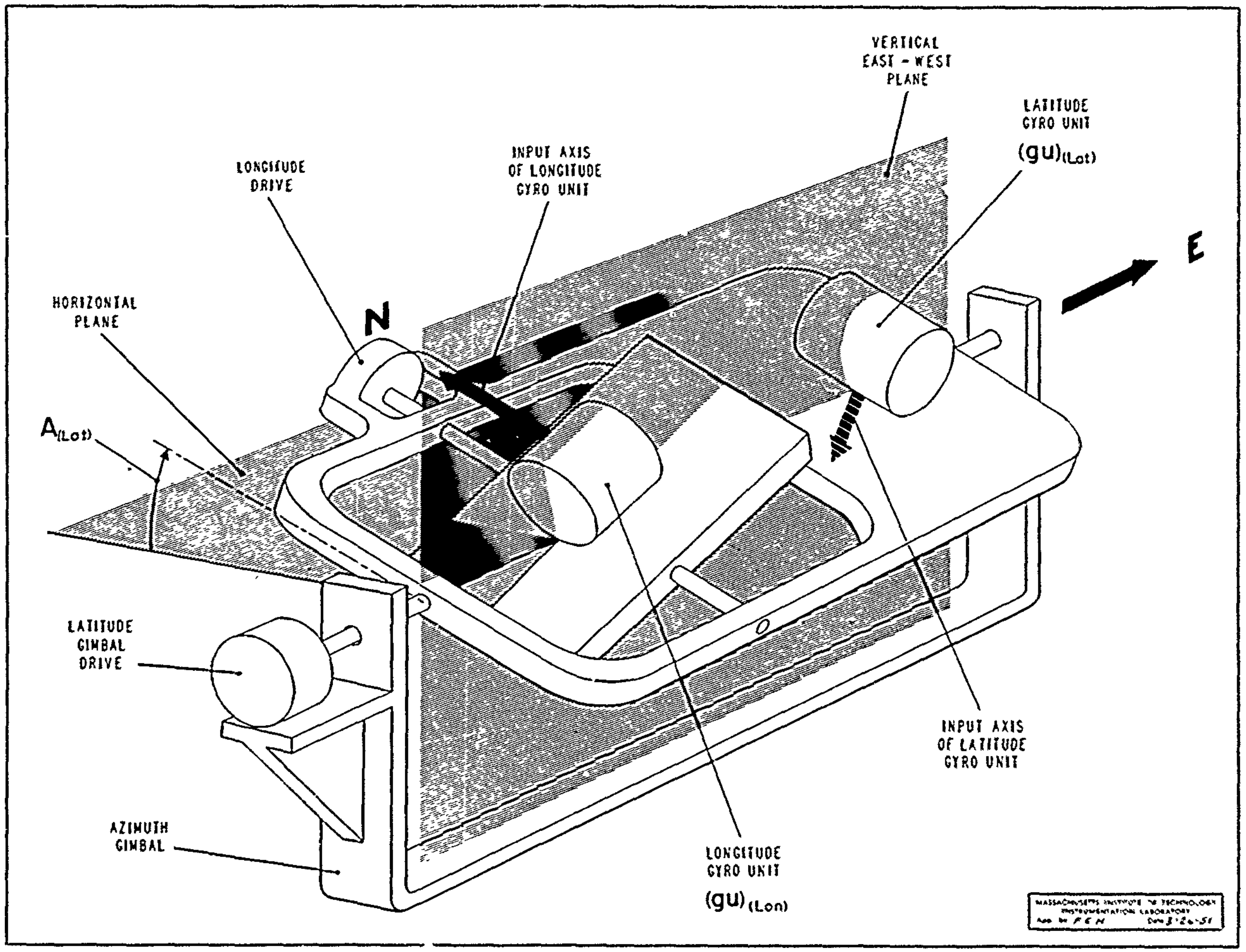


Figure 18. Latitude and Longitude Position Indicator Using the Geometric Integration of the Indicated Vertical.

The indicated vertical of such a single-axis system is given by the orientation of the outer gimbal in a two-gimbal system. The accelerometer is mounted on this gimbal. The gyro is mounted on the inner gimbal, where it maintains the reference vertical. Thus a reference vertical is physically "carried along" with the system as it moves over the Earth. The angle between the reference vertical and the indicated vertical is the indicated co-latitude. The reference gyro must be pre-aligned from external information with its input axis in the east - west direction and its output axis along the polar axis at the start of operations. The presence of an external pre-aligned reference obviates the need for a gyrocompass as an integral operating component in this system.

The extension of this system to the practical case, using three-axis stabilization, is not as simple as in the systems previously discussed. Three undisturbed (inertial) gyros are then required on the innermost gimbal, their input axes corresponding to mutually perpendicular x, y, and z axes respectively, with the z-axis oriented along the Earth's polar axis and the x and y axes arbitrarily oriented, but fixed in inertial space in a plane parallel to the equatorial plane (Fig. 20). The gimbal is then driven about the z-axis with a sidereal (clock) drive that removes the effect of the Earth's sidereal rotation from the inner gimbal. When the system changes longitude, indicated longitude rate is an input to the z-gyro unit. The z-gyro unit then generates a signal which produces an additional relative rotation between the inner gimbal and its support, at the negative of the indicated longitude rate. This causes the gyro input axes in the indicated equatorial plane to remain fixed in inertial space; i.e., they maintain equatorial "star lines".

A total of five gimbals is required for complete base motion isolation. Starting with the innermost gimbal, these are:

1. The inertial space gimbal, mounting the three inertial gyros.
2. The meridian gimbal, which supports the inertial space gimbal through a differential,so that the meridian gimbal can effectively be driven around the inertial space gimbal.
3. The azimuth gimbal, which is the controlled member, supporting two accelerometer units. Each accelerometer unit is a component in a loop similar to Fig. 19.
4. The pitch gimbal, which supports the azimuth gimbal and isolates the system from the pitch motion of the ship.
5. The roll gimbal, which supports the pitch gimbal and isolates the system from ship's roll.

The angle between the planes of the meridian and azimuth gimbals gives the indicated co-latitude. The angle between the inertial-space and meridian gimbals gives the indicated longitude difference. The performance equation for a single-axis version of this system is given in Derivation 10, where it is shown that the position-indicating ability of an ideal undamped system of this type is affected by the accuracy of the initial gyro-unit alignment.

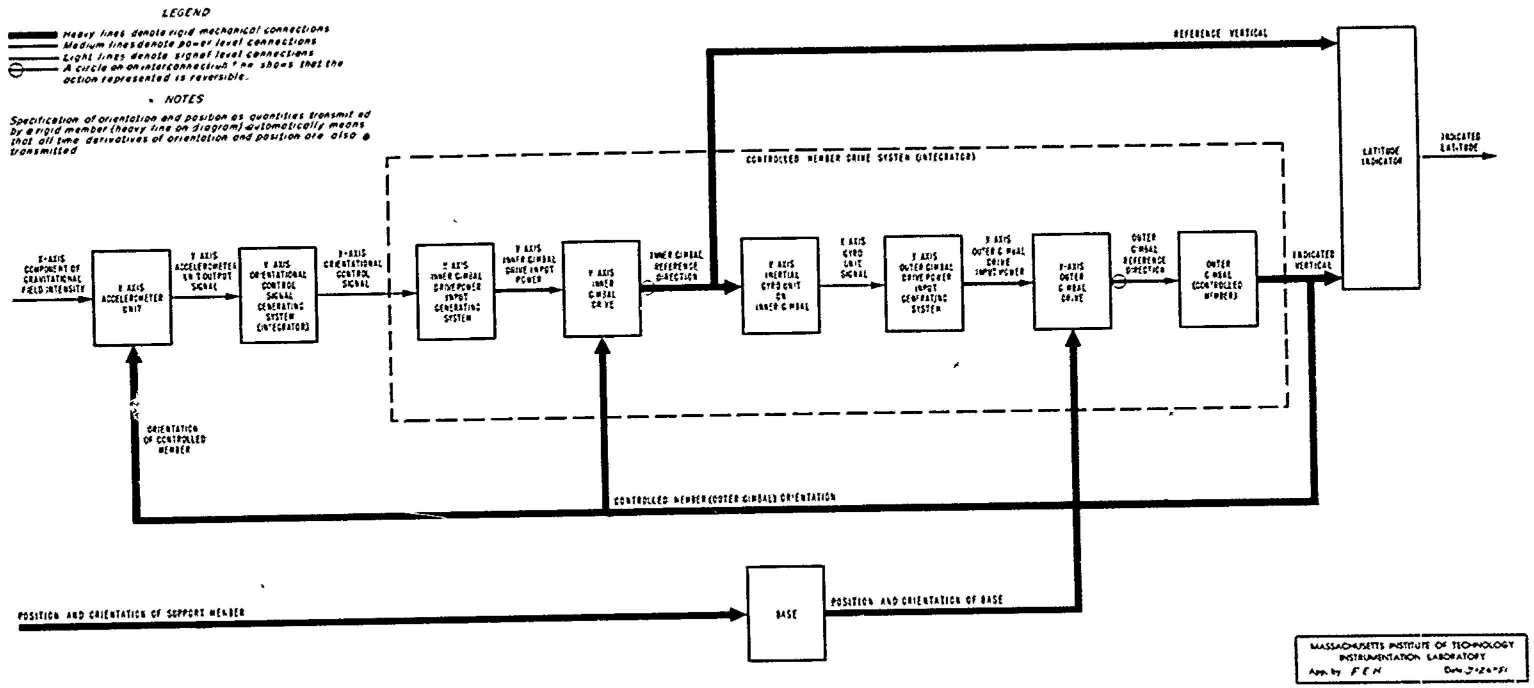


Figure 19. Functional Diagram Showing Interrelations Among Essential Operating Components of a Latitude Indicator Which Compares the Indicated Vertical with a Reference Vertical Maintained by an Inertial Gyro Unit.

Indication of Roll and Pitch

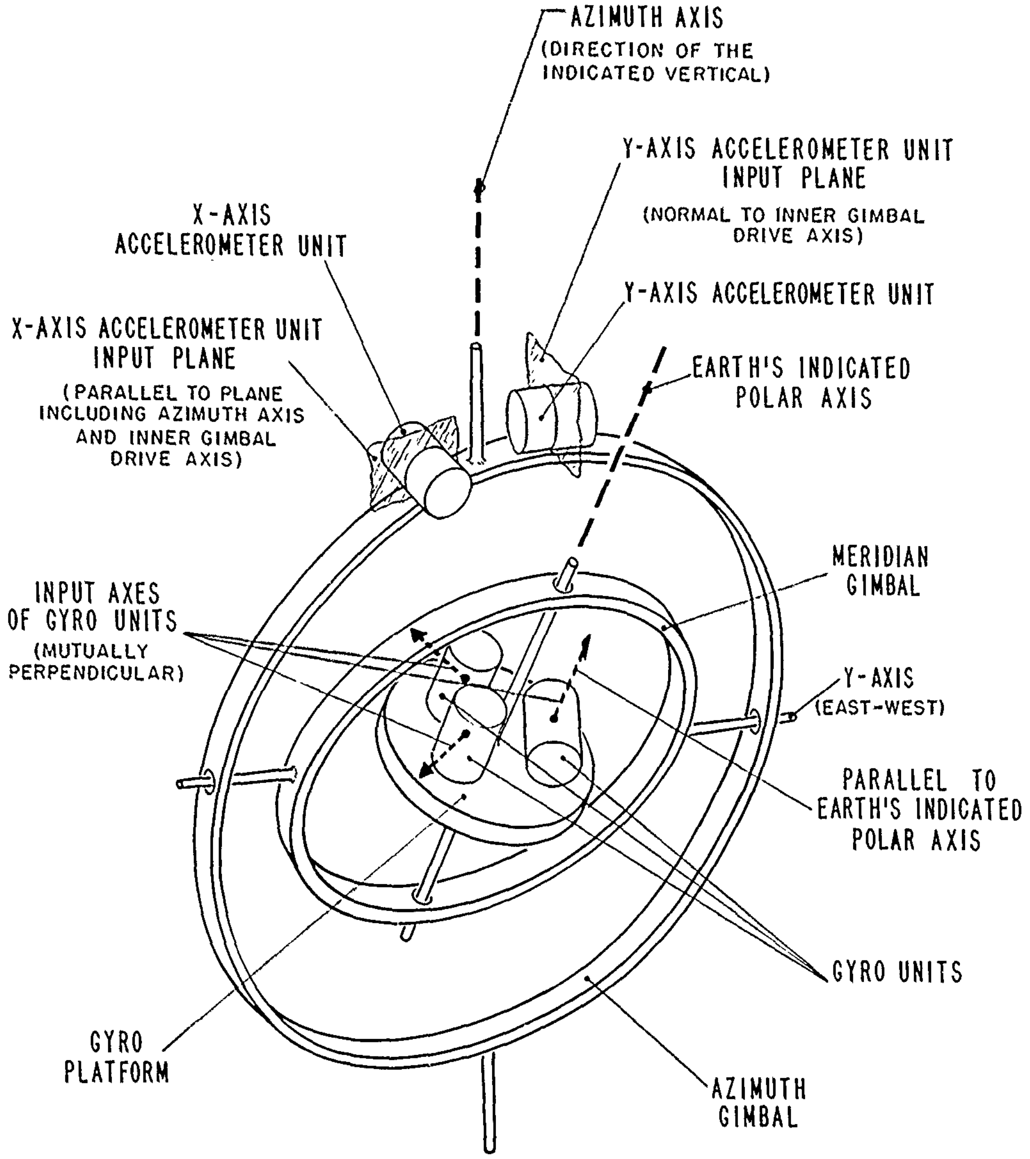
The functional diagram shown in Fig. 8-1, Derivation 8, shows a complete two-degree-of-freedom vertical indicator. This involves some components not yet discussed but nevertheless fundamental to all of the systems described — those using analytical as well as geometric integration. These components are associated with the mounting of the controlled member in each case and with the way the ship's motion influences the system. The latter concerns the indication of roll, pitch, and yaw; only roll and pitch will be taken up here, and yaw will be discussed with the gyrocompass later.

In Fig. 8-1, the ship orientation enters the gimbal support. Inside this support are, in order of decreasing size, the roll gimbal, the pitch gimbal, and the azimuth gimbal. The azimuth gimbal corresponds to the stable platform previously discussed. The fact that the accelerometers and tracking gyro units are mounted on the azimuth gimbal, although they generate signals which drive the roll and pitch gimbals, necessitates the resolvers and the so-called secant computer shown in Fig. 8-1. It should be emphasized that this geometric roll and pitch indication is applicable to all of the systems already discussed.

The Gyrocompass

The note in the lower left corner of Fig. 9 states that "for three-degree-of-freedom stabilization and orientational control, the z-axis gyro unit must be supplied a proper orientational control signal from a source not shown in this diagram". This source will necessarily be a compass; for the present purpose, a gyrocompass is indicated. The problem in orientation here is considerably simplified once the vertical-indication problem has been solved independently. Standard gyrocompass methods will, in general, be less useful than the one to be described, which takes full advantage of the horizontal stability of the controlled member which is to be stabilized in azimuth.

The Earth makes one revolution about its axis with respect to the "fixed stars" every sidereal day (23 hours, 56 minutes, 4.1 seconds of standard solar time). Hence, it possesses an angular velocity with respect to inertial space. At any point on the surface of the Earth, this angular velocity has components which are directed along the local vertical and in the horizontal plane directed north. Figure 21 shows these components as they exist at any point on the Earth's surface. Consider the special case where the system of Fig. 9 is at rest with respect to the Earth, and is settled, as far as its vertical-indicating function is concerned. Suppose, however, that the input axis of the y-gyro unit is not pointed east, so that the x-axis reference line on the controlled member does not, in consequence, indicate true north. The angle between indicated and true north can then be defined as the correction to indicated north. In operation, this angle will always be small, since the function of the azimuth tracking system is to cause the controlled member to align its x-axis with true north. When the system is at rest on the Earth and has an azimuth error, there is an angular velocity component input to the y-gyro unit. This input is specifically the horizontal component of Earth-rate, projected on the y-axis, i.e., multiplied by the sine of the correction



NOTE: Roll and pitch gimbals are not shown.

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Figure 20. Diagram Showing Essential Components of a Three-Axis Position Indicator Using an Inertial Reference.

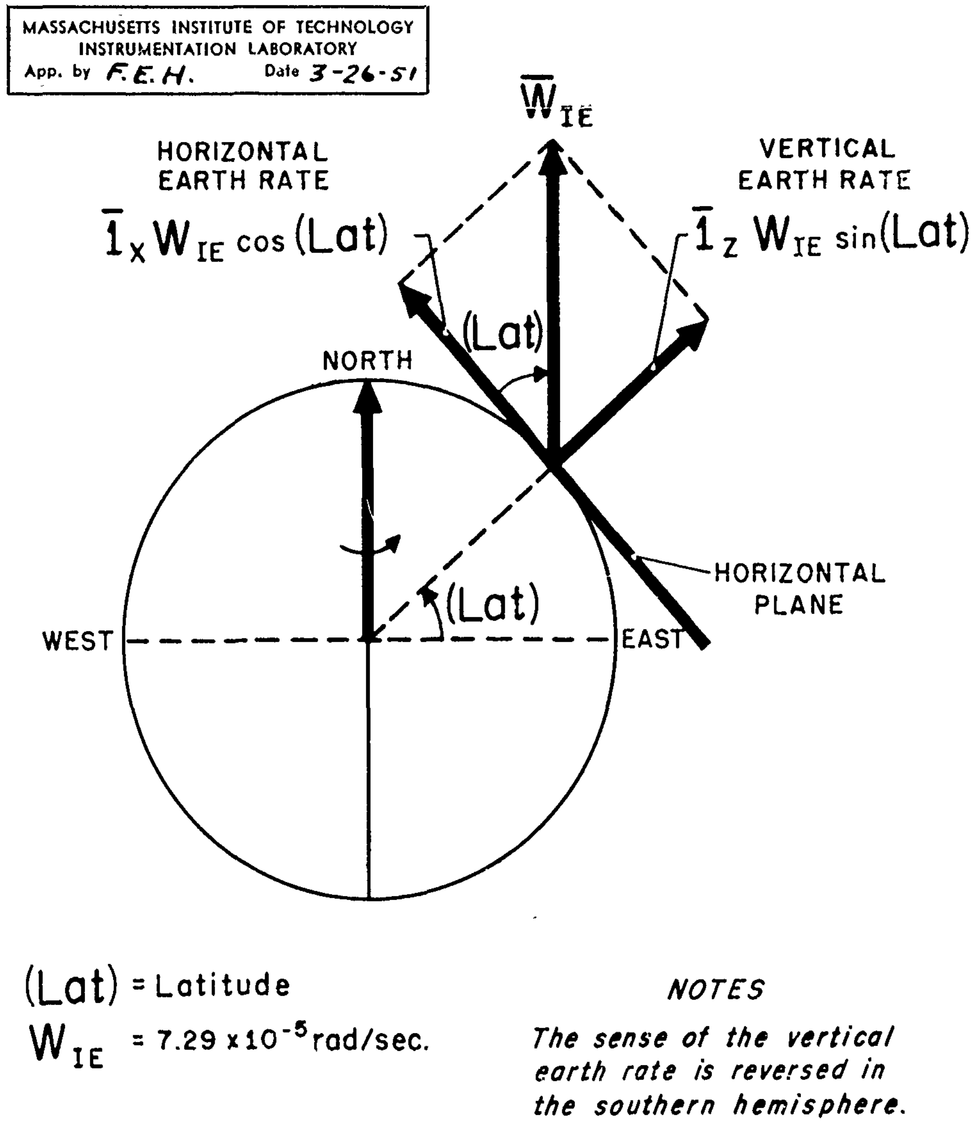

Figure 21. The Earth's Angular Velocity Components Which Affect a Gyrocompass.

to indicated north (Fig. 22). This input affects the y-axis base motion isolation system (Fig. 12) in such a manner as to cause the x-axis to dip when this axis is pointing west of north, i.e., the y-axis system is attempting to stabilize the controlled member in inertial space. This dip action causes the y-axis pendulum or accelerometer unit to send an orientational control signal to the y-gyro unit, thereby precessing the controlled member relative to inertial space in order to keep the controlled member horizontal. Since this orien-

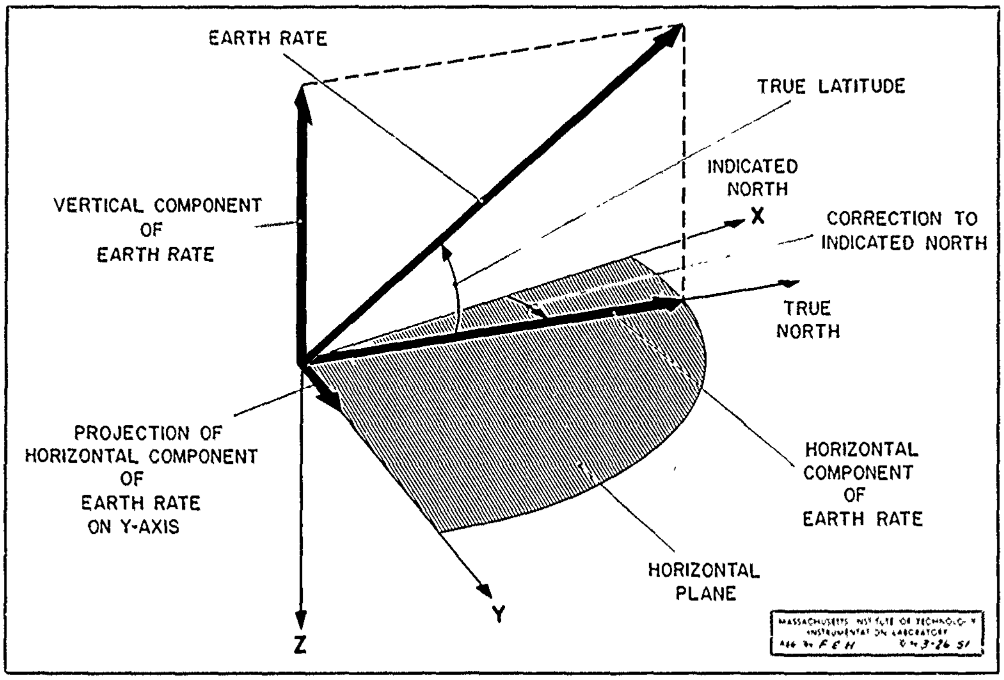

Figure 22. Horizontal Earth-Rate Component Projection on Y-Axis Due to Azimuth Misalignment.

tational control signal is proportional to the angular velocity projection on the y-axis, it is a measure of the azimuth error.

Figure 23 shows this signal as an input to the z-axis orientational control signal generating system. The z-axis controlled member drive system, on the basis of this Earth-rate signal from the y-gyro unit, then orients the controlled member in azimuth so that the correction to indicated north tends toward zero; i.e., so that the y-gyro unit Earth-rate component input is reduced to zero. However, as noted, the vertical indicator is simultaneously influenced by this input. The y-axis portion of the vertical indicator and the z-axis system are coupled in such a way that their dynamic stability characteristics are mutually interdependent. It is shown in Derivation 7 that their performance equations are similar, and that an azimuth disturbance will create required corrections to the indicated vertical and to indicated north which are approximately Schuler periodic.

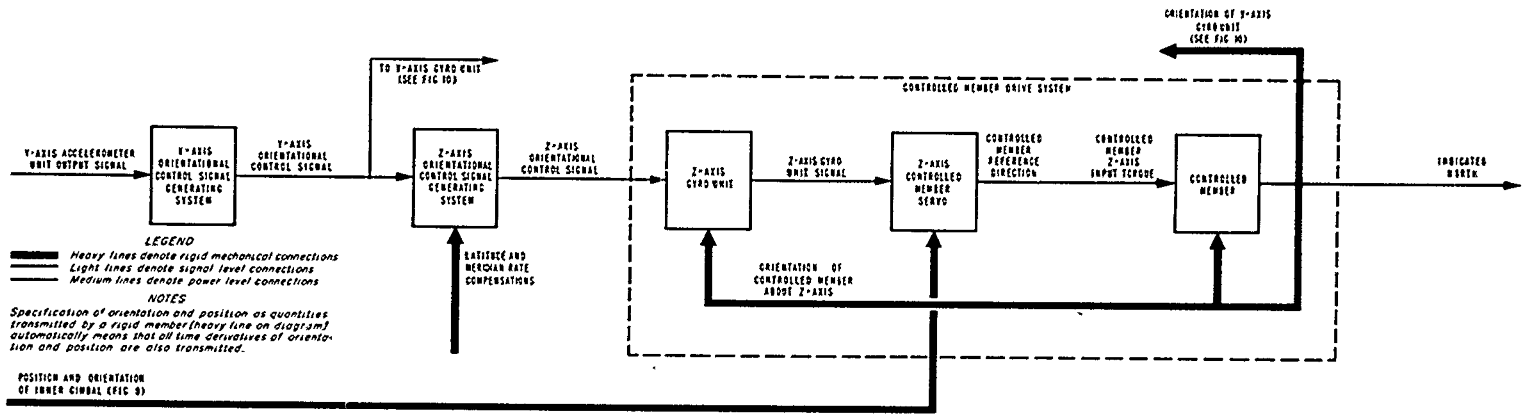

FIG 23 PARTIAL FUNCTIONAL SCHEMATIC DIAGRAM OF AN AZIMUTH STABILIZATION SYSTEM APPLICABLE TO THE Z-AXIS ORIENTATION OF THE CONTROLLED MEMBER IN FIG 9.

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Figure 23. Partial Functional Schematic Diagram of an Azimuth Stabilization System Applicable to the Z-Axis Orientation of the Controlled Member of Figure 9.

It should be noted that a separate rate gyro unit, with its input axis along the y-axis, rather than the y-gyro unit of Fig. 9, would accomplish the same purpose. However, the additional gyro unit would be unnecessary, since the y-gyro unit is already required for vertical indication, and the use of a separate rate gyro unit would still impose simultaneous stability conditions on the y- and z-loops.

Suppose that the system of Fig. 9 moves with respect to the Earth, and that the z-axis gyro unit of Fig. 9 is a component of the azimuth stabilizing loop of Fig. 23. The latitude rate of the system now appears in the y-axis orientational control signal as a term that is undesired in the z-axis system. Latitude rate can be effectively removed by the introduction of the ship's northward velocity component (from an external source) into the z-axis orientational control signal generating system (Fig. 23). (This correction is similar to those required in standard gyrocompasses.)

Change of longitude involves an associated rotation of the meridian about the vertical axis, due to the convergence of the meridian grid poleward (this effect is zero at the equator). Longitude rate effects can be compensated by injecting the x-axis orientational control signal, modified by the tangent of latitude, into the z-axis orientational control signal generating system. If the x-signal has in addition been compensated independently for Earth-rate, the z-gyro unit, operating with respect to inertial space, will also be so compensated. The role of the z-axis orientational control signal generating system in further processing of the y-axis orientational control signal is detailed in Derivation 7.

The combined system described thus performs the dual function of tracking the true vertical and true north.

SUMMARY

The self-contained inertial navigation systems discussed in this report have been considered only in ideal form, on the basis of the theoretical background. Imperfections due to actual equipment are deferred for the next report. Accordingly, no attempt is made here to find a most feasible system; this also is left to the next report.

APPENDIX

The mathematical derivations on the following pages are referred to by number in the foregoing text. The self-defining notation used in the equations is based on a formulation given by C. S. Draper, Notes on Instrument Engineering, M.I.T. Instrumentation Laboratory, Cambridge, Mass., September, 1950.

DERIVATION 1

SPECIFIC FORCE COMPONENTS ASSOCIATED WITH INERTIAL NAVIGATION

The object of this derivation is to obtain the specific force components associated with the motion of a point P representing the position of a vehicle in a navigation problem. The point is located, as shown in Figure 1-1, at a vector distance $\bar{R}_{mP}$ from the origin m of moving coordinate system m. Origin m is located at a vector distance $\bar{R}_{rm}$ from the origin r of reference

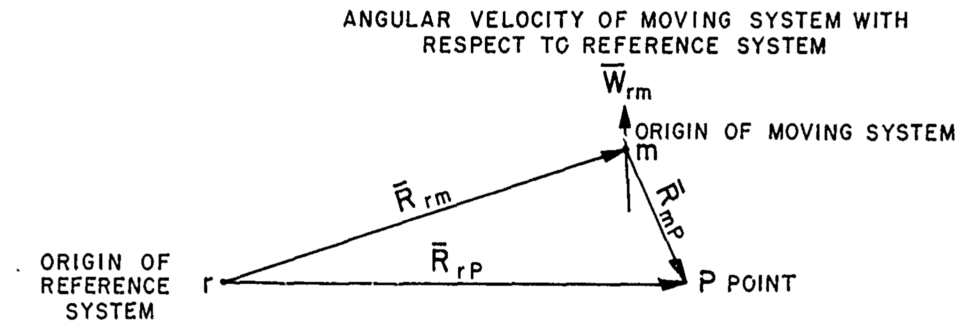

Figure 1-1. Geometrical Relationships of Location Vectors in Two Coordinate Systems.

coordinate system r. System m has motion of both translation and rotation with respect to system r. A gravitational field of intensity $\bar{G}$ is present at point P.

The vector distance $\bar{R}_{rP}$ from origin r to point P is

$$\bar{R}_{rP} = \bar{R}_{rm} + \bar{R}_{mP} \tag{1-1}$$

Differentiation of eq. (1-1) with respect to system r gives

$$[\dot{\bar{R}}_{rP}]_r = [\dot{\bar{R}}_{rm}]_r + [\dot{\bar{R}}_{mP}]_r \tag{1-2}$$

where $[\quad]_r$ denotes differentiation with respect to system r, and $(\dot{\ })$ denotes d/dt. Application of the equation of Coriolis to $[R_{mP}]_r$ gives

$$[\dot{\bar{R}}_{mP}]_r = [\dot{\bar{R}}_{mP}]_m + \bar{W}_{rm} \times \bar{R}_{mP} \tag{1-3}$$

where $\overline{W}_{rm}$ is the angular velocity of system m with respect to system r. Substitution of this relationship into eq. (1-2) gives

$$[\dot{\overline{R}}_{rP}]_r = [\dot{\overline{R}}_{rm}]_r + [\dot{\overline{R}}_{mP}]_m + \overline{W}_{rm} \times \overline{R}_{mP} \qquad (1\text{-}4)$$

In words, eq. (1-4) states

velocity of point P with respect to system r	=	velocity of point m with respect to system r	+	velocity of point P with respect to system m	+	cross product of angular velocity of system m with respect to system r and distance of point P from point m

Differentiation of eq. (1-4) with respect to system r gives

$$[\ddot{\overline{R}}_{rP}]_r = [\ddot{\overline{R}}_{rm}]_r + \left\{\overset{\bullet}{\overline{[\dot{\overline{R}}_{mP}]_m}}\right\}_r + \dot{\overline{W}}_{rm} \times \overline{R}_{mP} + \overline{W}_{rm} \times [\dot{\overline{R}}_{mP}]_r \qquad (1\text{-}5)$$

Application of the equation of Coriolis to $\left\{\overset{\bullet}{\overline{[\dot{\overline{R}}_{mP}]_m}}\right\}_r$ gives

$$\left\{\overset{\bullet}{\overline{[\dot{\overline{R}}_{mP}]_m}}\right\}_r = [\ddot{\overline{R}}_{mP}]_m + \overline{W}_{rm} \times [\dot{\overline{R}}_{mP}]_m \qquad (1\text{-}6)$$

Substitution of eqs. (1-3) and (1-6) into eq. (1-5) gives

$$[\ddot{\overline{R}}_{rP}]_r = [\ddot{\overline{R}}_{rm}]_r + [\ddot{\overline{R}}_{mP}]_m + \overline{W}_{rm} \times [\overline{W}_{rm} \times \overline{R}_{mP}] + \dot{\overline{W}}_{rm} \times \overline{R}_{mP} + 2\,\overline{W}_{rm} \times [\dot{\overline{R}}_{mP}]_m \qquad (1\text{-}7)$$

In words, eq. (1-7) states

acceleration of point P with respect to system r	=	acceleration of point m with respect to system r	+	acceleration of point P with respect to system m	+	centripetal acceleration of point P due to angular velocity $\overline{W}_{rm}$ of system m with respect to system r

+	tangential acceleration of point P due to angular acceleration $\dot{\overline{W}}_{rm}$ of system m with respect to system r	+	Coriolis acceleration of point P

For application of true position P_t with respect to the Earth let

Point r in eq. (1-7) ⟶ E, i.e., center of Earth
Point m in eq. (1-7) ⟶ E, i.e., center of Earth
System r in eq. (1-7) ⟶ E, i.e., Earth reference axes
System m in eq. (1-7) ⟶ P_t, i.e.. true position axes

Then eq. (1-7) becomes

$$[\ddot{\bar{R}}_{EP_t}]_E = [\ddot{\bar{R}}_{EP_t}]_{P_t} + \bar{W}_{EP_t} \times [\bar{W}_{EP_t} \times \bar{R}_{EP_t}] + \dot{\bar{W}}_{EP_t} \times \bar{R}_{EP_t}$$
$$+ 2\bar{W}_{EP_t} \times [\dot{\bar{R}}_{EP_t}]_{P_t} \qquad (1\text{-}8)$$

since $[\ddot{\bar{R}}_{EE}]_E = 0$.

For application to the acceleration of true position P_t with respect to inertial space, via the Earth, let

Point r in eq. (1-7) ⟶ E, i.e., center of Earth
Point m in eq. (1-7) ⟶ E, i.e., center of Earth
System r in eq. (1-7) ⟶ I, i.e., inertial space reference axes
System m in eq. (1-7) ⟶ E, i.e., Earth reference axes

Then eq. (1-7) becomes, using the relationship of eq. (1-3),

$$[\ddot{\bar{R}}_{EP_t}]_I = [\ddot{\bar{R}}_{EP_t}]_E + \bar{W}_{IE} \times [\bar{W}_{IE} \times \bar{R}_{EP_t}] + 2\bar{W}_{IE} \times [\dot{\bar{R}}_{EP_t}]_{P_t}$$
$$+ 2\bar{W}_{IE} \times [\bar{W}_{EP_t} \times \bar{R}_{EP_t}] \qquad (1\text{-}9)$$

since $[\ddot{\bar{R}}_{EE}]_I = 0$ and $\dot{\bar{W}}_{IE} = 0$.

Substitution of eq. (1-8) into eq. (1-9) gives

$$[\ddot{\bar{R}}_{EP_t}]_I = [\ddot{\bar{R}}_{EP_t}]_{P_t} + \bar{W}_{EP_t} \times [\bar{W}_{EP_t} \times \bar{R}_{EP_t}] + \dot{\bar{W}}_{EP_t} \times \bar{R}_{EP_t}$$
$$+ 2\bar{W}_{EP_t} \times [\dot{\bar{R}}_{EP_t}]_{P_t} + \bar{W}_{IE} \times [\bar{W}_{IE} \times \bar{R}_{EP_t}] + 2\bar{W}_{IE} \times [\dot{\bar{R}}_{EP_t}]_{P_t}$$
$$+ 2\bar{W}_{IE} \times [\bar{W}_{EP_t} \times \bar{R}_{EP_t}] \qquad (1\text{-}10)$$

In marine operation, where radial acceleration of the vehicle is small compared to gravitational acceleration, and the radial velocity of the vehicle is small compared to its surface velocity, $[\ddot{\bar{R}}_{EP_t}]_{P_t}$ and $[\dot{\bar{R}}_{EP_t}]_{P_t}$ may be neglected; eq. (1-10) then reduces to

$$[\ddot{\bar{R}}_E]_I = \bar{W}_{EP} \times [\bar{W}_{EP} \times \bar{R}_E] + \dot{\bar{W}}_{EP} \times \bar{R}_E + \bar{W}_{IE} \times [\bar{W}_{IE} \times \bar{R}_E]$$

$$+ 2\,\bar{W}_{IE} \times [\bar{W}_{EP} \times \bar{R}_E] \tag{1-11}$$

where $R_{EP_t} \longrightarrow R_E$ and the subscript t is dropped for convenience, since no ambiguity is possible by the simplification.

Expressed in terms of the associated inertia reaction specific force terms, eq. (1-11) becomes

$$(\bar{sf})_{(IR)} = (\bar{sf})_{(IR)W_{EP}} + (\bar{sf})_{(IR)\dot{W}_{EP}} + (\bar{sf})_{(IR)W_{IE}} + (\bar{sf})_{(IR)C} \tag{1-12}$$

where

$(\bar{sf})_{(IR)} \quad = -[\ddot{\bar{R}}_E]_I$

$(\bar{sf})_{(IR)W_{EP}} \quad = -\bar{W}_{EP} \times [\bar{W}_{EP} \times \bar{R}_E] = -\bar{W}_{EP} \times \bar{v}_{EP}$; negligible compared to $(\bar{sf})_G$ below, at marine speeds.

$(\bar{sf})_{(IR)\dot{W}_{EP}} \quad = -\dot{\bar{W}}_{EP} \times \bar{R}_E$

$(\bar{sf})_{(IR)W_{IE}} \quad = -\bar{W}_{IE} \times [\bar{W}_{IE} \times \bar{R}_E]$; effect of Earth's daily rotation.

$(\bar{sf})_{(IR)C} \quad = -2\,\bar{W}_{IE} \times [\bar{W}_{EP} \times \bar{R}_E] = -2\,\bar{W}_{IE} \times \bar{v}_{EP}$; horizontal transverse component of Coriolis acceleration; negligible* at marine speeds.

---

* The magnitude of the tangential component of $(\bar{sf})_{(IR)C}$ is

$$\left| 2\bar{W}_{IE} \times \bar{v}_{EP} \right|_{(tang)} = 2 v_{EP} W_{IE} \sin(\text{Lat})$$

For Lat = 45° and v_{EP} = 10 knots, the error in true vertical indication attributable to the

When the gravitational specific force is included, eq. (1-12) becomes

$$(\overline{sf})_{(res)} = (\overline{sf})_G + (\overline{sf})_{(IR)} \tag{1-13}$$

$$= (\overline{sf})_g + \left[(\overline{sf})_{(IR)} - (\overline{sf})_{(IR)W_{IE}} \right] \tag{1-14}$$

where

$(\overline{sf})_{(res)}$ = resultant specific force

$(\overline{sf})_G$ = $\overline{G}$ = gravitational field intensity

$(\overline{sf})_g$ = $(\overline{sf})_G + (\overline{sf})_{(IR)W_{IE}} = \overline{g}$ = gravity field intensity

It is to be noted that

$(\overline{sf})_g$ is vertical, by definition

$(\overline{sf})_{(IR)W_{EP}}$ is vertical

$(\overline{sf})_{(IR)\dot{W}_{EP}}$ is horizontal

$(\overline{sf})_{(IR)C}$ is horizontal and transverse

The vector $(\overline{sf})_{(res)}$ is the quantity to which a pendulous element responds.

---

tangential Coriolis specific force is

$$[(E)V_t]_C \doteq \frac{(sf)_{(IR)C}}{g} \cong \frac{1}{5} \text{ minute of arc}$$

This is the reason $(sf)_{(IR)C}$ is neglected in the detailed system discussions to follow.

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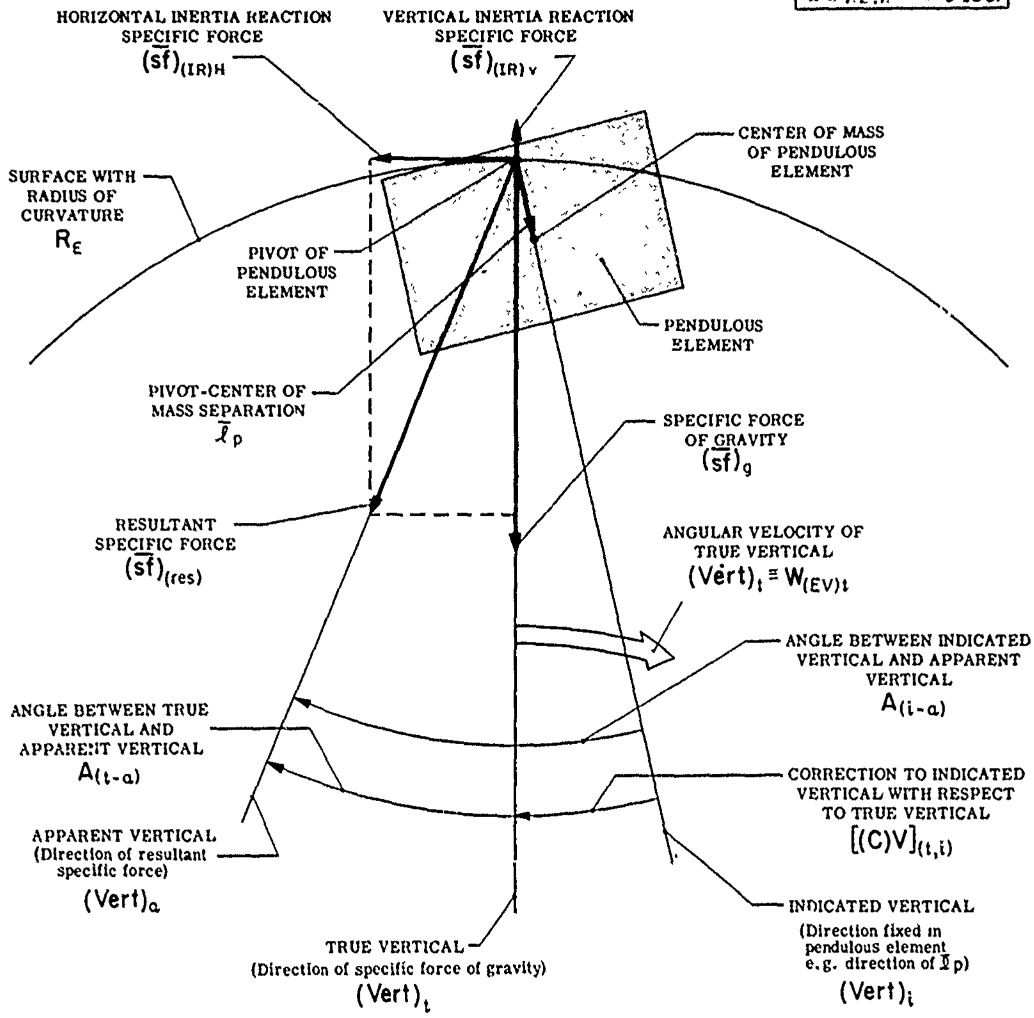


Geometrical Relationships

$$\sin A_{(t-a)} = \sin\left\{[(C)V]_{(t,i)} + A_{(t-a)}\right\} = \sin[(C)V]_{(t,i)} \cos A_{(t-a)} + \cos[(C)V]_{(t,i)} \sin A_{(t-a)}$$

$$\sin A_{(t-a)} = \frac{(sf)_{(IR)H}}{(sf)_{(res)}} \qquad (sf)_{(IR)H} = R_E(\ddot{Vert})_t$$

$$(sf)_{(IR)V} = -R_E(\dot{Vert})_t^2$$

$$\cos A_{(t-a)} = \frac{(sf)_g + (sf)_{(IR)V}}{(sf)_{(res)}} \qquad (sf)_g = g$$

Note that the inertia reaction specific force associated with the centripetal acceleration due to the Earth's daily rotation is included in the specific force of gravity.

Figure 2-1. Geometrical Relationships Associated with a Pendulous Element as an Indicator of the Vertical.

DERIVATION 2

CHARACTERISTICS OF A PENDULOUS ELEMENT AS AN INDICATOR OF THE VERTICAL

Newton's law of motion applied to a pendulous element that is tracking the resultant specific force is

$$m_p r_{gp}^2 (\ddot{Vert})_i = m_p l_p (sf)_{(res)} \sin A_{(i-a)} \qquad (2\text{-}1)$$

where

m_p = mass of pendulous element

r_{gp} = radius of gyration of pendulous element

For other definitions see Fig. 2-1.

Substitution in eq. (2-1) of the relationships for $(\ddot{Vert})_i$ and $(sf)_{(res)} \sin A_{(i-\varepsilon)}$ listed in Fig. 2-1 gives, with cancellation of m_p and division by r_{gp}^2,

$$(\ddot{Vert})_t - [(\ddot{C})V]_{(t,i)} = \frac{l_p}{r_{gp}^2} [(sf)_g + (sf)_{(IR)V}] \sin [(C)V]_{(t,i)}$$

$$+ \frac{l_p}{r_{gp}^2} (sf)_{(IR)H} \cos [(C)V]_{(t,i)} \qquad (2\text{-}2)$$

Substitution in eq. (2-2) of the relationships for the specific force components listed in Fig. 2-1, using the approximations* that $\sin [(C)V]_{(t,i)} \cong [(C)V]_{(t,i)}$ and $\cos [(C)V]_{(t,i)} \cong 1$, gives:

---

* With any pendulous element that is a suitable indicator of the vertical, the angle $[(C)V]_{t,i}$ should be only a few minutes of arc, or at most a few degrees.

$$(\dot{\text{Vert}})_t - [(C)V]_{(t,i)} = \frac{l_p}{r_{gp}^2}[g - R_E (\dot{\text{Vert}})^2_t][(C)V]_{(t,i)}$$

$$+\frac{l_p}{r_{gp}^2} R_E (\ddot{\text{Vert}})_t \qquad (2\text{-}3)$$

A regrouping of the terms in eq. (2-3) gives

$$[(C)V]_{(t,i)} + \frac{l_p}{r_{pg}^2}[g - R_E (\dot{\text{Vert}})^2_t][(C)V]_{(t,i)} = [1 - \frac{l_p}{r_{gp}^2} R_E](\ddot{\text{Vert}})_t \qquad (2\text{-}4)$$

The condition of Schuler tuning is that the coefficient on the right-hand side of eq. (2-4) be zero. In an attempt to accomplish this let

$$\frac{l_p}{r_{gp}^2} = \frac{1}{R_{E(set)}} \qquad (2\text{-}5)$$

Substitution of the relationship of eq. (2-5) into eq. (2-4) gives

$$[(C)V]_{(t,i)} + \frac{g - R_E (\dot{\text{Vert}})^2_t}{R_{E(set)}}[(C)V]_{(t,i)} = [1 - \frac{R_E}{R_{E(set)}}](\ddot{\text{Vert}})_t \qquad (2\text{-}6)$$

DERIVATION 3
VERTICAL INDICATION

A. Basic Principles

The vector angle representing the correction to the indicated vertical with respect to the true vertical is expressed mathematically as

$$[\overline{(C)V}]_{(t,\,i)} = \bar{1}_{V_i} \times \bar{1}_{V_t} \qquad (3\text{-}1)$$

where the quantities are defined in Fig. 3-1.

Indication with Respect to Earth Axes — Accelerometer Tracking

The rate of change of this angle with respect to indicated position axes, i.e., in a set of coordinates that consists of the indicated vertical and two other mutually orthogonal axes arbitrarily oriented in the indicated horizontal plane, is

$$[\dot{\overline{(C)V}}]_{(t,\,i)P_i} = \overline{W}_{(EV)t} - \overline{W}_{(EV)i} - \overline{W}_{(EP)V_i} \times [\overline{(C)V}]_{(t,\,i)} \qquad (3\text{-}2)$$

where $[\dot{\overline{(C)V}}]_{(t,\,i)P_i}$ = rate of change of indicated vertical correction with respect to indicated position axes

$\overline{W}_{(EV)t}$ = angular velocity of true vertical with respect to the Earth

$\overline{W}_{(EV)i}$ = angular velocity of indicated vertical with respect to the Earth

$\overline{W}_{(EP)V_i}$ = angular velocity of indicated position axes with respect to the Earth about the indicated vertical

The magnitude of the indicated horizontal component of the resultant specific forces, i.e., the quantity sensed by an accelerometer, is expressed mathematically in eq. (3-3).

$$(sf)_{(res)H_i} = \left| \overline{(sf)}_{(res)H_i} \right| = \left| \overline{1}_{V_i} \times \overline{(sf)}_{(res)} \right| \tag{3-3}$$

where

$(sf)_{(res)H_i}$ = indicated horizontal component of resultant specific force

$|\quad|$ = denotes magnitude of the enclosed vector quantity.

Let the system operate such that the angular velocity of the indicated vertical with respect to the Earth is a function of the magnitude of the indicated horizontal component of the resultant specific force and directed so as to rotate the indicated vertical toward the resultant specific force vector, namely

$$\overline{W}_{(EV)i} = (PF)_{(Vi)(f,W)} \; [\overline{1}_{V_i} \times \overline{(sf)}_{(res)}] \tag{3-4}$$

where

$(PF)_{(Vi)(f,W)}$ = vertical-indicating performance function for a specific force input and an angular velocity output.

Substitution of eq. (3-4) into eq. (3-2), plus the relationships for the resultant specific force from Derivation 1, gives

$$\dot{\overline{[(C)V]}}_{(t,i)P_i} = \overline{W}_{(EV)t} - (PF)_{(Vi)(f,W)} \, \overline{1}_{V_i} \times \Big\{ \overline{g} - (\dot{\overline{W}}_{EP})_E \times \overline{R}_E$$

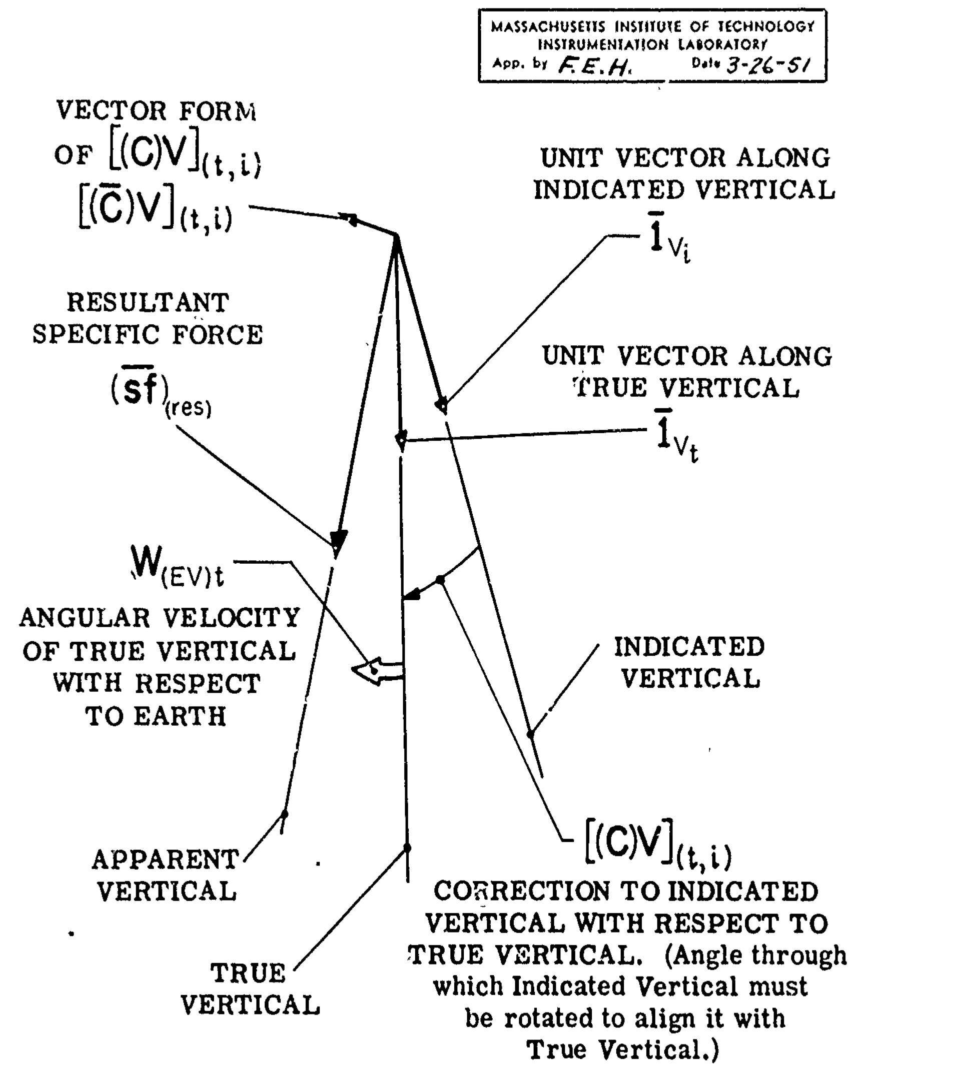
$$- \overline{W}_{EP} \times (\overline{W}_{EP} \times \overline{R}_E) - 2\overline{W}_{IE} \times (\overline{W}_{EP} \times \overline{R}_E) \Big\} - \overline{W}_{(EP)V_i} \times \overline{[(C)V]}_{(t,i)} \tag{3-5}$$

The specific force components in eq. (3-5) can be conveniently resolved into components parallel to the true vertical and perpendicular to the true vertical (the arbitrary nature of axes in the horizontal plane requires no further subdivision into components), namely

$$\overline{g} = \overline{1}_{V_t} g$$

$$\overline{R}_E = -\overline{1}_{V_t} R_E$$

$$\overline{W}_{EP} = \overline{1}_{V_t} W_{(EP)V_t} + \overline{W}_{(EV)t}$$



AS A VECTOR ANGLE

$$[(\bar{C})V]_{(t,i)} = \bar{1}_{V_i} \times 1_{V_t} \frac{[(C)V]_{(t,i)}}{\sin C_{t,i}} \cong \bar{1}_{V_i} \times \bar{1}_{V_t}, \text{ SINCE } [(C)V]_{(t,i)} \text{ IS SMALL}$$

Figure 3-1. Geometrical Relationships Associated with the Vector Angle Form of the Correction to the Indicated Vertical with Respect to the True Vertical.

Substitution of these relationships into eq. (3-5) gives the specific force in terms of components along the true vertical, along the angular velocity vector of the true vertical (which must be perpendicular to the true vertical) and perpendicular to both of these, namely

$$\overline{[(\dot{C})V]}_{(t,i)P_i} = \overline{W}_{(EV)t} - (PF)_{(Vi)(f,W)}\overline{1}_{V_i} \times \Big\{ \overline{1}_{V_t} [g - R_E W^2_{(EV)t}$$

$$- 2R_E(\overline{W}_{IE}) \cdot \overline{W}_{(EV)t})] + \overline{W}_{(EV)t} R_E [W_{(EP)V_t} + 2(\overline{W}_{IE} \cdot \overline{1}_{V_t})]$$

$$+ [(\dot{\overline{W}}_{EP})_E \times \overline{1}_{V_t}] R_E \Big\} - \overline{W}_{(EP)V_i} \times \overline{[(C)V]}_{(t,i)} \qquad (3\text{-}6)$$

Carrying out the vector operation indicated in eq. (3-6), regrouping terms and omitting the components parallel to the indicated vertical as not being applicable to the indication of the vertical, gives

$$\overline{[(\dot{C})V]}_{(t,i)P_i} + (PF)_{(Vi)(f,W)}\,(sf)_{(res)V_t}\,\overline{[(C)V]}_{(t,i)} = \overline{W}_{(EV)t}$$

$$-(PF)_{(Vi)(f,W)} R_E (\dot{\overline{W}}_{(EV)t})_{P_i} - (PF)_{(Vi)(f,W)} R_E [W_{(EP)V_t}$$

$$+2(\overline{W}_{IE} \cdot \overline{1}_{V_t})]\,(\overline{1}_{V_i} \times \overline{W}_{(EV)t}) - \overline{W}_{(EP)V_i} \times \overline{[(C)V]}_{(t,i)} \qquad (3\text{-}7)$$

where

$$(sf)_{(res)V_t} = g - R_E W^2_{(EV)t} - 2R_E(\overline{W}_{IE} \cdot \overline{W}_{(EV)t}) \qquad (3\text{-}8)$$

= component of resultant specific force along the true vertical

Let $(PF)_{(Vi)(f,W)}$ be the following operator:

$$(PF)_{(Vi)(f,W)} = S_{(Vi)(f,\dot{W})} \int_{P_i} (\quad)\,dt \qquad (3\text{-}9)$$

where $\int_{P_i}$ denotes integration of a function of the indicated axes with respect to the time. Substitution of equation (3-9) into eq. (3-7) gives eq. (3-10):

$$\dot{\overline{[(C)V]}}_{(t,i)P_i} + S_{(Vi)(f,\dot{W})} \int_{P_i} (sf)_{(res)V_t} \overline{[(C)V]}_{(t,i)}\, dt = \overline{W}_{(EV)t}$$

$$- S_{(Vi)(f,\dot{W})} \int_{P_i} R_E (\dot{\overline{W}}_{(EV)t})_{P_i}\, dt - S_{(Vi)(f,\dot{W})} \int_{P_i} R_E [\overline{W}_{(EP)V_t}$$

$$+ 2(\overline{W}_{IE} \cdot \overline{1}_{V_t})](\overline{1}_{V_i} \times \overline{W}_{(EV)t})\, dt - \overline{W}_{(EP)V_i} \times \overline{[(C)V]}_{(t,i)} \qquad (3\text{-}10)$$

Differentiating (3-10) with respect to the time gives the differential performance equation in terms of functions of the indicated axes:

$$\ddot{\overline{[(C)V]}}_{(t,i)P_i} + S_{(Vi)(f,\dot{W})} (sf)_{(res)V_t} \overline{[(C)V]}_{(t,i)}$$

$$= (1 - S_{(Vi)(f,\dot{W})} R_E)(\dot{\overline{W}}_{(EV)t})_{P_i} - S_{(Vi)(f,\dot{W})} R_E [\overline{W}_{(EP)V_t}$$

$$+ 2(\overline{W}_{IE} \cdot \overline{1}_{V_t})](\overline{1}_{V_i} \times \overline{W}_{(EV)t}) - \overset{\bullet}{\overbrace{(\overline{W}_{(EP)V_i} \times \overline{[(C)V]}_{(t,i)})}}_{P_i} \qquad (3\text{-}11)$$

where it is assumed that variations in R_E are slow enough to be treated as quasi-static and variations in $(sf)_{(res)V_t}$ are either slow enough to be quasi-static or fast enough to be ineffective.

For Schuler tuning let

$$S_{(Vi)(f,\dot{W})} = \frac{1}{R_S}$$

where R_S is a set value made as nearly equal as possible to the magnitude of R_E. If it is then assumed that $R_S \doteq R_E$, eq. (3-11) becomes

$$\ddot{\overline{[(C)V]}}_{(t,i)P_i} + \frac{(sf)_{(res)V_t}}{R_S} \overline{[(C)V]}_{(t,i)} = -[\overline{W}_{(EP)V_t} + 2(\overline{W}_{(IE)} \cdot \overline{1}_{V_t})](\overline{1}_{V_i}$$

$$\times \overline{W}_{(EV)t}) - (\overline{W}_{(EP)V_i} \times \overline{[(C)V]}_{(t,i)})_{P_i} \qquad (3\text{-}12)$$

The terms $\overline{W}_{(EP)V_t}$ $(\overline{1}_{V_i} \times \overline{W}_{(EV)t})$, and $2(\overline{W}_{IE} \cdot \overline{1}_{V_t})(\overline{1}_{V_i} \times W_{(EV)t})$ are small in comparison with $\frac{(sf)_{(res)V_t}}{R_S}$, and can be neglected. This can be shown as follows:

$W_{(EP)V_t}$, the magnitude of the rate of change of the meridian about the true vertical with respect to the Earth, is given by

$$W_{(EP)V_t} = \frac{v_B}{R_E} \sin(Co) \tan(Lat) \tag{3-13}$$

where $\quad v_B$ = Velocity of base

Co = course of base

Lat = latitude

$$W_{(EV)t} = \frac{v_B}{R_E} \tag{3-14}$$

$$\left| W_{(EP)V_t} (\bar{1}_{V_i} \times \overline{W}_{(EV)t}) \right| \cong W_{(EP)V_t} W_{(EV)t} \tag{3-15}$$

and

$$\left| 2 (\overline{W}_{IE} \cdot \bar{1}_{V_t})(\bar{1}_{V_i} \times \overline{W}_{(EV)t}) \right| \cong 2\, W_{(EV)t} W_{IE} \sin(Lat)$$

Assume that the base is traveling due east at 10 knots at 70° latitude:

$$\frac{W_{(EP)V_t} W_{(EV)t}}{\dfrac{(sf)_{(res)V_t}}{R_s}} = \frac{R_s W_{(EP)V_t} W_{(EV)t}}{(sf)_{(res)V_t}} = 00.02 \text{ min. of arc} \tag{3-16}$$

$$\frac{2\, W_{IE} \sin(Lat)\, W_{(EV)t}}{\dfrac{(sf)_{(res)V_t}}{R_s}} = \frac{2\, R_s\, W_{IE} \sin(Lat)\, W_{(EV)t}}{(sf)_{(res)V_t}} = 00.25 \text{ min of arc} \tag{3-17}$$

Neglecting the above terms, eq. (3-12) reduces to

$$\ddot{\overline{[(C)V]}}_{(t,i)P_i} + \frac{(sf)_{(res)V_t}}{R_s} \overline{[(C)V]}_{(t,i)} = -\overbrace{(\overline{W}_{(EP)V_i} \times [(C)V]_{(t,i)})}^{\cdot} \tag{3-18}$$

The term $(W_{(EP)V_i} \times [(C)V]_{(t,i)})$ represents an orientation transfer, not a magnitude effect, and represents a secondary coupling between the x and

y systems. Since this will only produce a "spiraling in" effect as the system approaches its null position, it will be ignored in subsequent discussions.

Equation (3-18) now reduces to

$$\ddot{\overline{[(C)V]}}_{(t,\,i)P_i} + \frac{^{(sf)}(res)\,V_t}{R_s}\,\overline{[(C)V]}_{(t,\,i)} = 0 \qquad (3\text{-}19)$$

B. Instrumentation of a Single-Axis System

<u>Definitions of Symbols Used in the Following Derivations</u>

a_c	Coriolis acceleration
a_{EB}	Acceleration of base with respect to Earth
a_{IB}	Acceleration of base with respect to inertial space
P_i	Indicated position (latitude or longitude) of vehicle
$(P_i)_o$	Indicated position of vehicle at start of problem
P_t	True position of base
$[(C)P]_{(t,\,i)}$	Position correction
$[(C)P]_{(t,\,i)o}$	Position correction at start of problem
$[(C)V]_{(t,\,i)}$	Correction to the indicated vertical
$[(C)V]_{(t,\,i)o}$	Correction to indicated vertical at start of problem
$[(C)V]_{(t,\,r)}$	Correction to reference vertical
$[(C)V]_{(t,\,r)o}$	Correction to reference vertical at start of problem
$e_{(\ )}$	Output signal voltage of device indicated by subscript
$\overline{G}$	Gravitational field vector
$\overline{g}$	Gravity field vector
H	Angular momentum of gyro unit rotor
$i_{(\ )}$	Output current of device indicated by subscript

M_g	Torque on gyro gimbal
$R = R_E$	Radius of the Earth, a function of geographic position
R_s	Set value of Earth radius used for Schuler tuning
$S_{(\ )(\ ,\ )}$	Sensitivity of component or component-group indicated by first subscript, with input and output indicated by second subscript
V_i	Indicated vertical
$(V_i)_o$	Indicated vertical at start of problem
V_r	Reference vertical
V_t	True vertical
$(V_t)_o$	True vertical at start of problem
$W_{(EV)i}$	Angular velocity of indicated vertical with respect to the Earth
$W_{(EV)r}$	Angular velocity of reference vertical with respect to the Earth
$W_{(EV)t}$	Angular velocity of true vertical with respect to the Earth
W_{IB}	Angular velocity of base with respect to inertial space
W_{IE}	Angular velocity of Earth with respect to inertial space
$W_{(IV)i}$	Angular velocity of indicated vertical with respect to inertial space
cmds	Controlled member drive system

Figure 3-2 is a mathematical functional diagram of a single-axis system designed to indicate the vertical under dynamic conditions. The component will be considered as ideal and the system to be undamped (non-ideal components and damped systems will be analyzed in the second report), and it will be shown that the equation of motion of the system is (3-19).

The quantity $[(C)V]_{(t,i)}$ is defined by equation (3-1); that is, $[(C)V]_{(t,i)}$ is the angle between the true vertical and the indicated vertical.

Then, in scalar form and operator notation

$$p[(C)V]_{(t,i)} = W_{(EV)t} - W_{(EV)i} \tag{3-20}$$

where $$p = \frac{d}{dt} \tag{3-21}$$

Equation (3-20) corresponds to eq. (3-2) with the coupling term $(\overline{W}_{(EP)Vi} \times [\overline{(C)V}]_{(t,i)})$ neglected.

The proposed system consists essentially of an accelerometer, an integrator, a single-degree-of-freedom integrating gyro unit, and a drive system for the stable platform or controlled member. The accelerometer is mounted on, and near the axis of rotation of, the controlled member. It will therefore sense the indicated acceleration of the base with respect to inertial space, or the projection of the resultant specific inertia reaction force on the indicated horizontal plane (in general this force will contain both acceleration and gravity effects).

The performance equation for the accelerometer unit is (neglecting accelerometer dynamics*):

$$e_{(au)} = S_{(au)(a,e)}(g[(C)V]_{(t,i)} + a_{IB})^{**} \tag{3-22}$$

where e_{au} = accelerometer unit output signal

$S_{(au)(a,e)}$ = accelerometer unit sensitivity for an acceleration input and a voltage output

a_{IB} = acceleration of the base with respect to inertial space

Since the system intrinsically minimizes $[(C)V]_{(t,i)}$, this angle will be small enough in general to permit the assumptions:

$$\sin [(C)V]_{(t,i)} \cong [(C)V]_{(t,i)}$$

$$\cos [(C)V]_{(t,i)} \cong 1$$

In Derivation 1 it was shown that, at submarine velocities

$$a_{IB} \cong RpW_{(EV)t} \tag{3-23}$$

---

* To incorporate Schuler tuning characteristics, the system is designed to have a natural period of approximately 84 minutes. Therefore, the dynamics associated with the accelerometer unit and the controlled member drive are neglected.

** In this scalar equation the centripetal acceleration due to the Earth's daily rotation is included in g rather than a_{IB}. The net vertical force is effectively g for submarine operation.

Substituting this in (3-22) gives

$$e_{(au)} = S_{(au)(a,e)} \left(g[(C)V]_{(t,i)} + RpW_{(EV)t} \right) \qquad (3\text{-}24)$$

The integrator performance equation is given by

$$i_{(int)} = S_{(int)(e,\dot{i})} \frac{1}{p} e_{(au)} + i_{(int)o} \qquad (3\text{-}25)$$

$i_{(int)}$ = current output of integrator

$S_{(int)(e,\dot{i})}$ = sensitivity of integrator for voltage input, current rate output

$i_{(int)o}$ = integrator current at start of problem

The primary purpose of the gyro unit is to provide a means of stabilizing the controlled member against arbitrary motions of the base; however, in the present analysis, the gyro unit is simply considered to be part of the angular velocity generating system. The input current produces a torque on the gyro unit which is balanced by the angular velocity of the controlled member with respect to inertial space, thereby keeping the gyro unit on null,

$$S_{(tg)(i,M)}[i_{(int)} + i_W] = HW_{(cmds)} = HW_{(IV)i} \qquad (3\text{-}26)$$

where

$S_{(tg)(i,M)}$ = sensitivity of gyro unit torque generator for current input and torque output

i_W = current compensating for rotation of earth with respect to inertial space

H = angular momentum of gyro unit

$W_{(cmds)}$ = angular velocity of controlled member with respect to inertial space

$W_{(IV)i} \equiv W_{(cmds)}$ = angular velocity of indicated vertical with respect to inertial space

Let

$$S_{(cmds)(i,W)} = \frac{S_{(tg)(i,M)}}{H} \qquad (3\text{-}27)$$

where

$S_{(cmds)(i,W)}$ = sensitivity of controlled member drive system for a current input and an angular velocity output

From the foregoing,

$$W_{(IV)i} = W_{(EV)i} + W_{IE} = S_{(cmds)(i,W)} i_{(int)} + S_{(cmds)(i,W)} i_W \qquad (3\text{-}28)$$

Set

$$i_W = \frac{W_{IE}}{S_{(cmds)(i,W)}} \qquad (3\text{-}29)$$

then $$W_{(EV)i} = S_{(cmds)(i,W)} i_{(int)} \qquad (3\text{-}30)$$

$$W_{(EV)i} = S_{(au)(a,e)} S_{(int)(e,\dot{i})} S_{(cmds)(i,W)} \frac{1}{p} (g[(C)V]_{(t,i)} + RpW_{(EV)t})$$

$$+ S_{(cmds)(i,W)} i_{(int)o} \qquad (3\text{-}31)$$

Equation (3-31) corresponds to equation (3-4).

Substituting eqs. (3-31) into (3-20) and differentiating the result,

$$[p^2 + S_{(au)(a,e)} S_{(int)(e,\dot{i})} S_{(cmds)(i,W)} g] [(C)V]_{(t,i)}$$

$$= (1 - S_{(au)(a,e)} S_{(int)(e,\dot{i})} S_{(cmds)(i,W)} R) p W_{(EV)t} \qquad (3\text{-}32)$$

For Schuler tuning, set

$$S_{(au)(a,e)} S_{(int)(e,\dot{i})} S_{(cmds)(i,W)} = \frac{1}{R_s} \qquad (3\text{-}33)$$

Eq. (3-32) reduces to

$$(p^2 + \frac{g}{R_s}) [(C)V]_{(t,i)} = 0 \qquad (3\text{-}34)$$

and is identical with (3-19).

The solution to (3-34) is

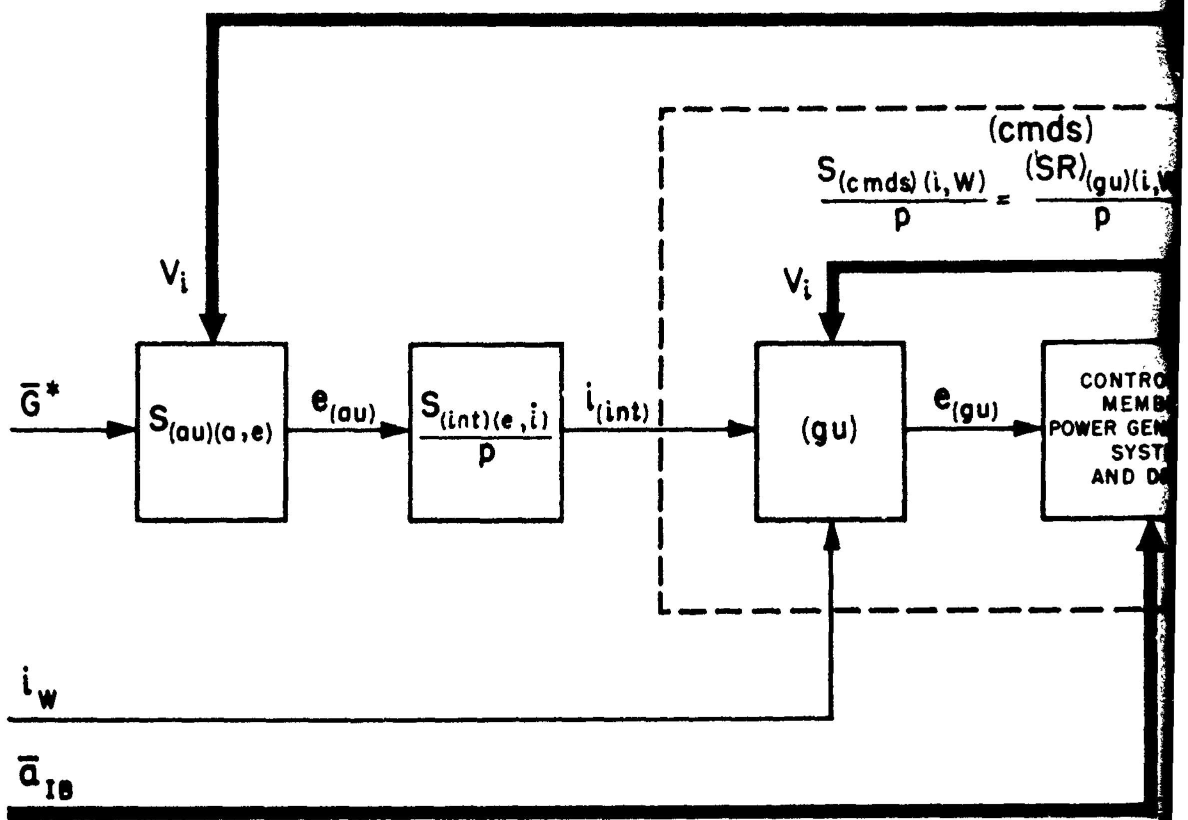
$$[(C)V]_{(t,i)} = [(C)V]_{(t,i)(o)} \cos\sqrt{\frac{g}{R_s}}\, t + [(\dot{C})V]_{(t,i)o} \sqrt{\frac{R_s}{g}} \sin\sqrt{\frac{g}{R_s}}\, t \qquad (3\text{-}35)$$

where

$[(C)V]_{(t,i)o}$ = Correction to indicated vertical at start of problem

$[(\dot{C})V]_{(t,i)o}$ = Rate of change of correction to indicated vertical at start of problem.

NOTE: $(SR)_{(gu)(i,W)}$ *is defined as the* *sensitivity ratio* *of the gyro unit, i.e., the ratio of the sensitivit*
for a current input-torque output to the sensitivity for an angular velocity input-torque output, the
torques being balanced to null by the gyro unit operation. The controlled member drive is sufficiently
to make $S_{(cmds)(i,W)} = (SR)_{(gu)(i,W)}$, *as in any high-gain feedback loop.*



Figu

gyro unit, i.e., the ratio of the sensitivity
angular velocity input-torque output, these
e controlled member drive is sufficiently fast
ain feedback loop.

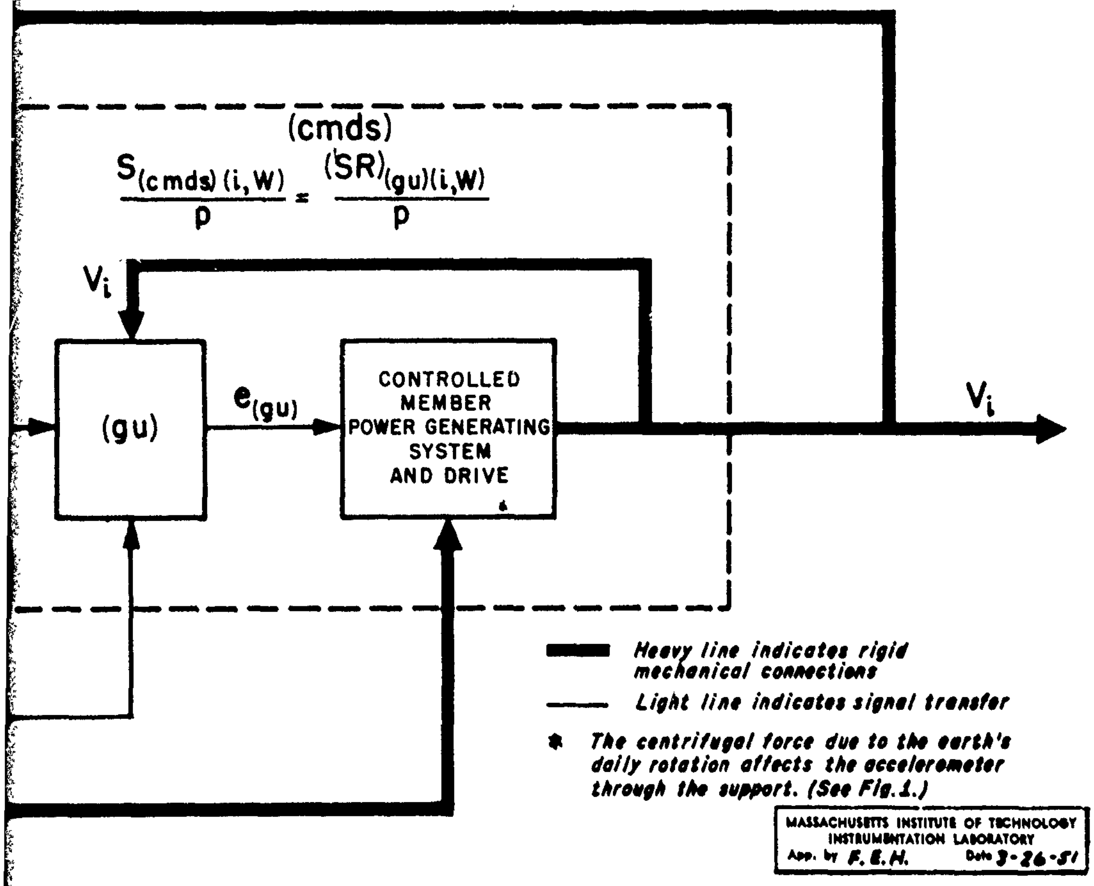


Figure 3-2. Mathematical Functional Diagram of a Basic Single-Axis Vertical Indicating System Using the Components Shown in Figure 15.

3

DERIVATION 4

POSITION INDICATION BY MEANS OF OPEN-CHAIN INTEGRATION OF THE ANGULAR VELOCITY OF THE INDICATED VERTICAL

Figure 4-1 is a mathematical functional diagram of a single-axis system for the indication of position. Here, the input to the gyro unit in Derivation 3B is integrated once to give the indicated position. The gyro unit input is proportional to the angular velocity of the indicated vertical. Thus, the indicated position is given by:

$$P_i = S_{(Pi)(i,\dot{P})} \frac{1}{p} i_{(int)} + (P_i)_o \tag{4-1}$$

where $(P_i)_o$ is the indicated position at the start of the problem.

From (3-30), $i_{(int)}$ is related to the angular velocity of the indicated vertical as follows:

$$i_{(int)} = \frac{W_{(EV)i}}{S_{(cmds)(i,W)}} \qquad [3\text{-}30]$$

and from (3-21)

$$p[(C)V]_{(t,i)} = W_{(EV)t} - W_{(EV)i} \qquad [3\text{-}20]$$

$$P_i = S_{(Pi)(i,\dot{P})} \frac{1}{p} \frac{W_{(EV)t}}{S_{(cmds)(i,W)}} - S_{(Pi)(i,\dot{P})} \frac{1}{p} \frac{p[(C)V]_{(t,i)}}{S_{(cmds)(i,W)}} + (P_i)_o \tag{4-2}$$

Set the magnitude of the position integrator sensitivity equal to that of the controlled member drive system sensitivity:

$$S_{(Pi)(i,\dot{P})} = S_{(cmds)(i,W)} \tag{4-3}$$

(This constitutes an arbitrary adjustment of the system parameters)

Note that

$$\frac{1}{p} W_{(EV)t} = P_t - (P_t)_o \tag{4-4}$$

and

$$\frac{1}{p}\, p[(C)V]_{(t,i)} = [(C)V]_{(t,i)} - [(C)V]_{(t,i)o} \tag{4-5}$$

(4-2) now becomes

$$P_i = P_t - (P_t)_o - [(C)V]_{(t,i)} + [(C)V]_{(t,i)o} + (P_i)_o \tag{4-6}$$

Define the quantity $[(C)P]_{(t,i)}$ as the correction to the indicated position:

$$[(C)P]_{(t,i)} \equiv P_t - P_i \tag{4-7}$$

$$[(C)P]_{(t,i)o} = (P_t)_o - (P_i)_o \tag{4-8}$$

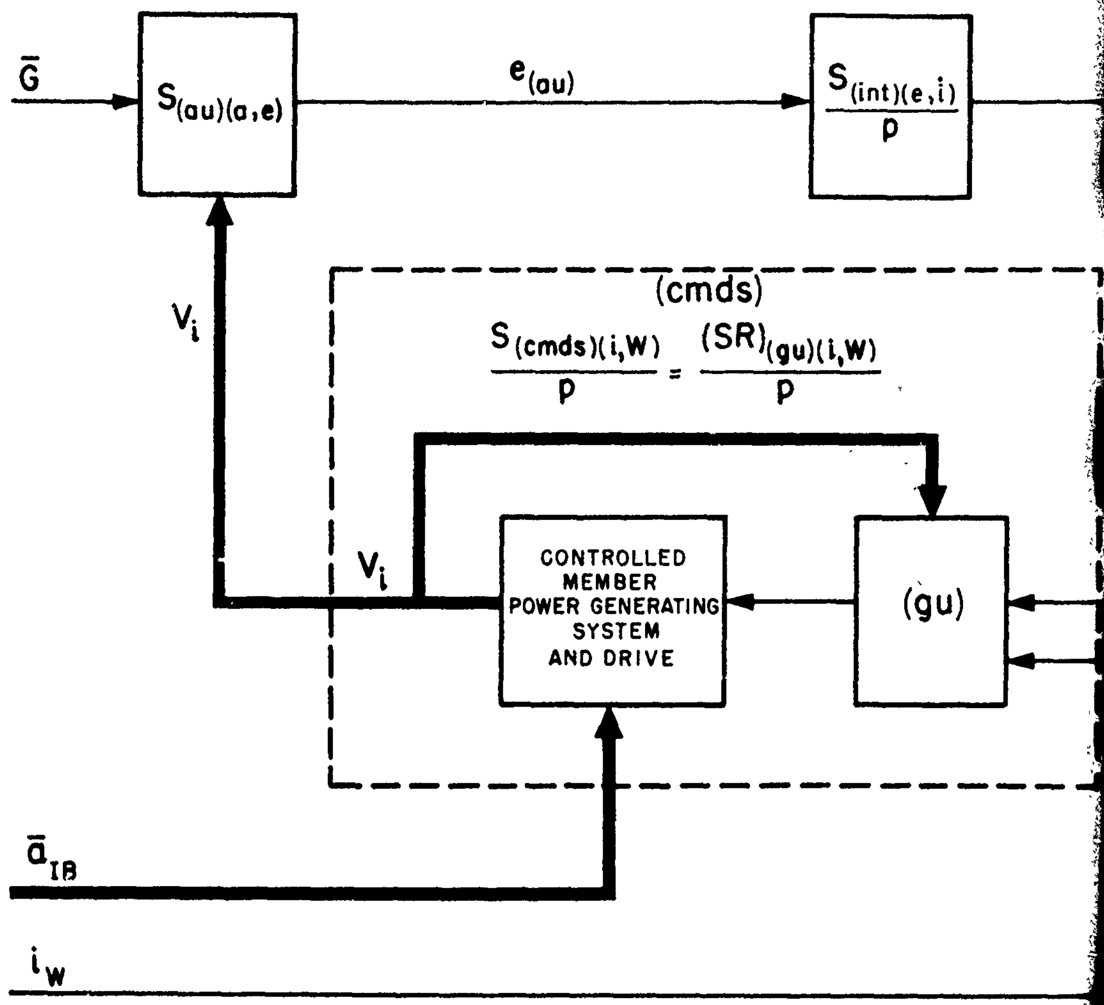
$[(C)V]_{(t,i)}$ may be evaluated from eq. (3-35), Derivation 3:

$$[(C)V]_{(t,i)} = [(C)V]_{(t,i)o} \cos\sqrt{\frac{g}{R_s}}\,t + [(C)\dot{V}]_{(t,i)o}\sqrt{\frac{R_s}{g}} \sin\sqrt{\frac{g}{R_s}}\,t \qquad [3\text{-}35]$$

From the above

$$[(C)P]_{(t,i)} = [(C)P]_{(t,i)o} - [(C)V]_{(t,i)o}\left(1 - \cos\sqrt{\frac{g}{R_s}}\,t\right)$$

$$+ [(C)\dot{V}]_{(t,i)o}\sqrt{\frac{R_s}{g}} \sin\sqrt{\frac{g}{R_s}}\,t \tag{4-9}$$



Ḡ
S(au)(a,e)
e(au)
S(int)(e,i)/p
Vi
(cmds)
S(cmds)(i,W)/p = (SR)(gu)(i,W)/p
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CONTROLLED MEMBER POWER GENERATING SYSTEM AND DRIVE
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1

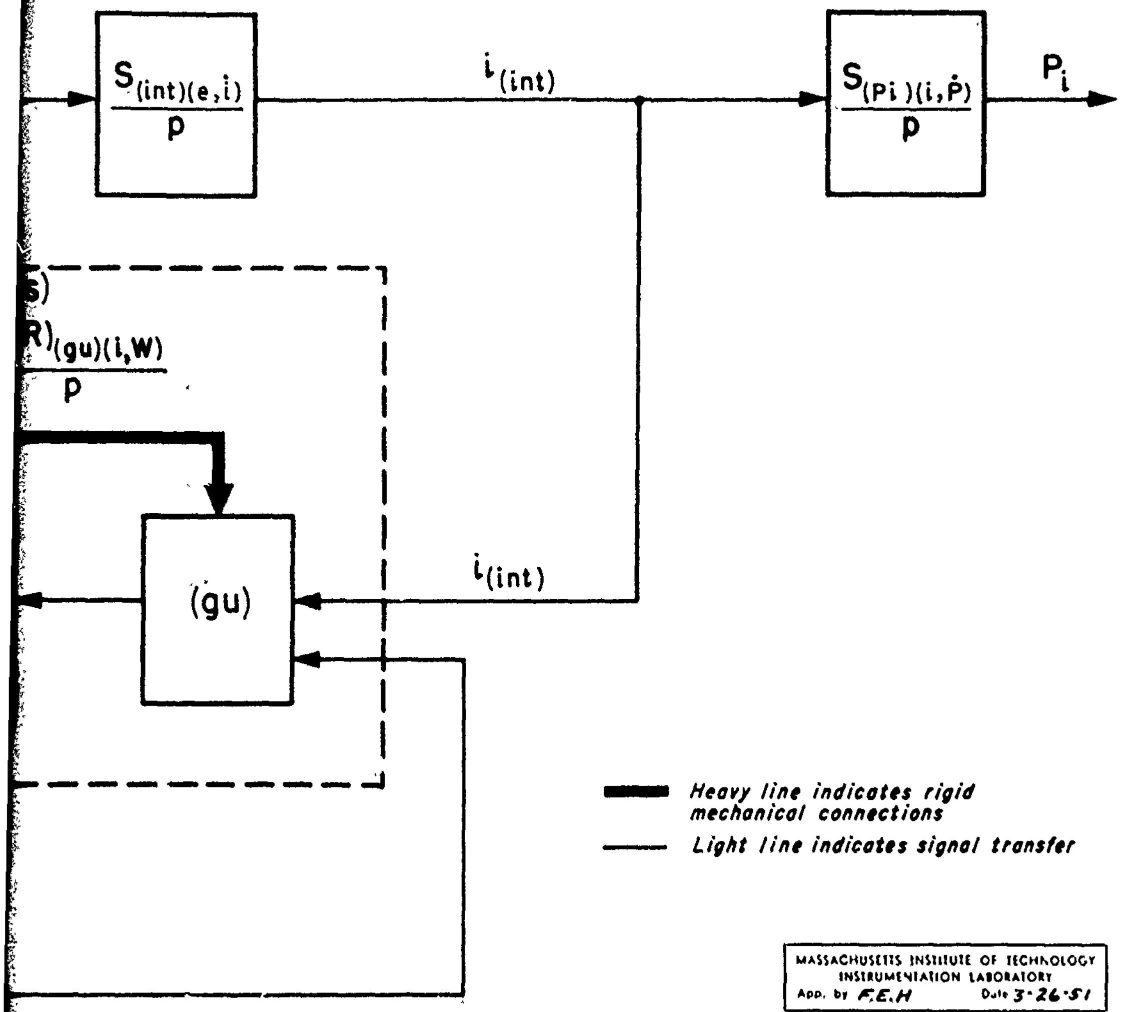


Figure 4-1. Mathematical Functional Diagram of a System for Indicating Position by Means of an Open-Chain Integration of the Angular Velocity of the Indicated Vertical.

DERIVATION 5

POSITION INDICATION BY MEANS OF OPEN-CHAIN INTEGRATION OF THE ANGULAR ACCELERATION OF THE INDICATED VERTICAL

Figure 5-1 is a mathematical functional diagram of another system for position indication. Here, the accelerometer unit output in the basic vertical indicator is integrated twice with respect to the time to give the indicated position. The accelerometer unit output is proportional to the angular acceleration of the indicated vertical; this is discussed in the foregoing text. The indicated position, from Fig. 5-1, is given by:

$$P_i = S_{(Pdi)(e,\ddot{P})} \frac{1}{p^2} e_{(au)} + (\dot{P}_i)_o t + (P_i)_o \qquad (5\text{-}1)$$

From (3-22) and (3-23) in Derivation 3 the accelerometer unit output is:

$$e_{(au)} = S_{(au)(a,e)} (g[(C)V]_{(t,i)} + RpW_{(EV)t}) \qquad (5\text{-}2)$$

Adjust the system parameters so that

$$S_{(Pdi)(e,\ddot{P})} = \frac{1}{R_s S_{(au)(a,e)}} \qquad (5\text{-}3)$$

$$P_i = \frac{1}{p^2} \frac{g}{R_s} [(C)V]_{(t,i)} + \frac{1}{p^2} pW_{(EV)t} + (\dot{P}_i)_o t + (P_i)_o \qquad (5\text{-}4)$$

From (3-34)

$$(p^2 + \frac{g}{R_s}) [(C)V]_{(t,i)} = 0 \qquad [3\text{-}34]$$

$$\frac{1}{p^2} p^2 [(C)V]_{(t,i)} = [(C)V]_{(t,i)} - [(C)V]_{(t,i)o} - [(\dot{C})V]_{(t,i)o} t \qquad (5\text{-}5)$$

$$\frac{1}{p^2} pW_{(EV)t} = P_t - (P_t)_o - (\dot{P}_t)_o t \qquad (5\text{-}6)$$

$$P_i = -[(C)V]_{(t,i)} + [(C)V]_{(t,i)o} + [(\dot{C})V]_{(t,i)o}t + P_t - (P_t)_o$$

$$- (\dot{P}_t)_o t + (\dot{P}_i)_o t + (P_i)_o \qquad (5\text{-}7)$$

The correction to the indicated position, from (4-7) in Derivation 4, is

$$[(C)P]_{(t,i)} = P_t - P_i \qquad [4\text{-}7]$$

Also

$$[(C)P]_{(t,i)o} = (P_t)_o - (P_i)_o \qquad [4\text{-}8]$$

$$[(\dot{C})P]_{(t,i)o} = (\dot{P}_t)_o - (\dot{P}_i)_o \qquad (5\text{-}8)$$

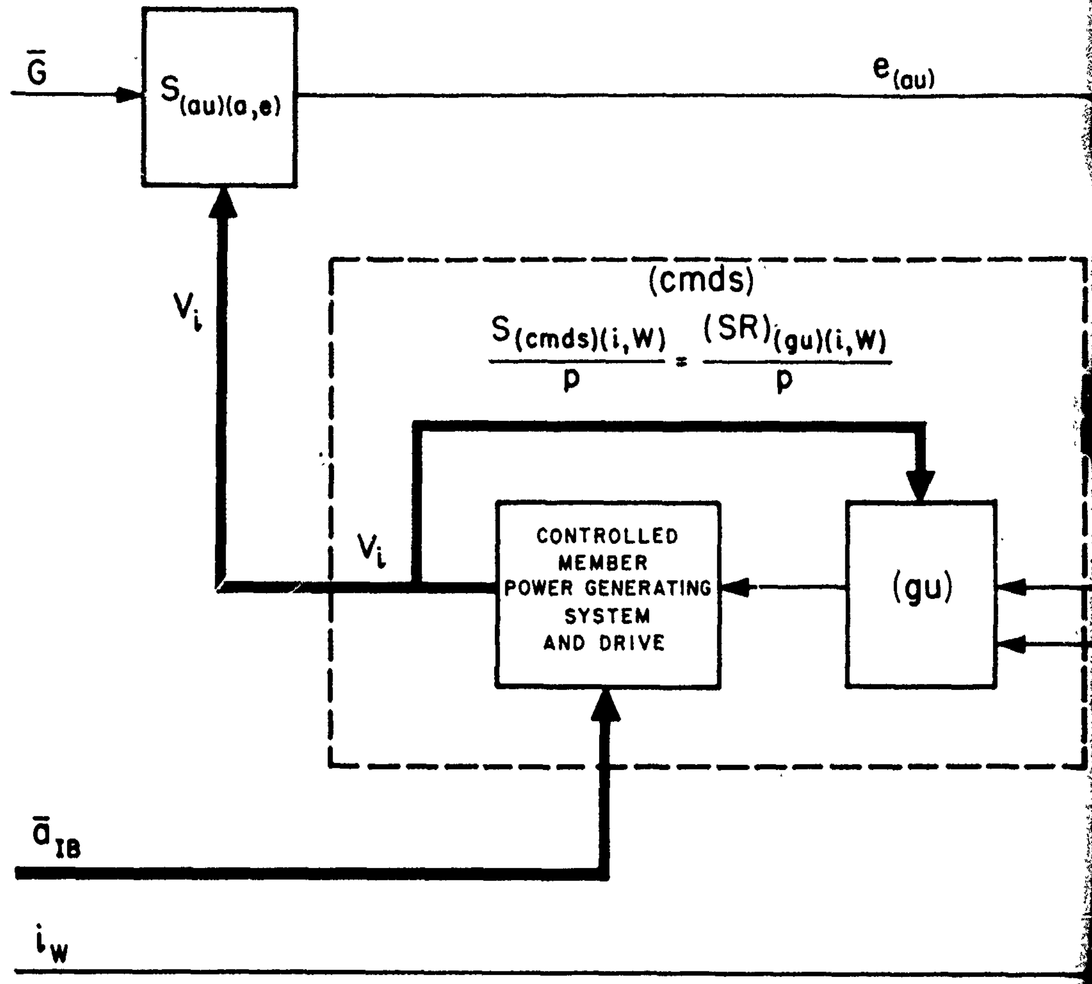
From eq. (3-35) in Derivation 3

$$[(C)V]_{(t,i)} = [(C)V]_{(t,i)o} \cos\sqrt{\frac{g}{R_s}}\, t + [(\dot{C})V]_{(t,i)o}\sqrt{\frac{R_s}{g}}\, \sin\sqrt{\frac{g}{R_s}}\, t \qquad [3\text{-}35]$$

Substituting the above into (4-7) gives

$$[(C)P]_{(t,i)} = [(C)P]_{(t,i)o} + [(\dot{C})P]_{(t,i)o}t - [(C)V]_{(t,i)o}\left(1 - \cos\sqrt{\frac{g}{R_s}}\, t\right)$$

$$- [(\dot{C})V]_{(t,i)o}\left[t - \sqrt{\frac{R_s}{g}}\, \sin\sqrt{\frac{g}{R_s}}\, t\right] \qquad (5\text{-}9)$$



$\bar{G}$
$S_{(au)(a,e)}$
$e_{(au)}$
V_i
(cmds)
$\frac{S_{(cmds)(i,W)}}{p} = \frac{(SR)_{(gu)(i,W)}}{p}$
V_i
CONTROLLED MEMBER POWER GENERATING SYSTEM AND DRIVE
(gu)
$\bar{a}_{IB}$
i_W

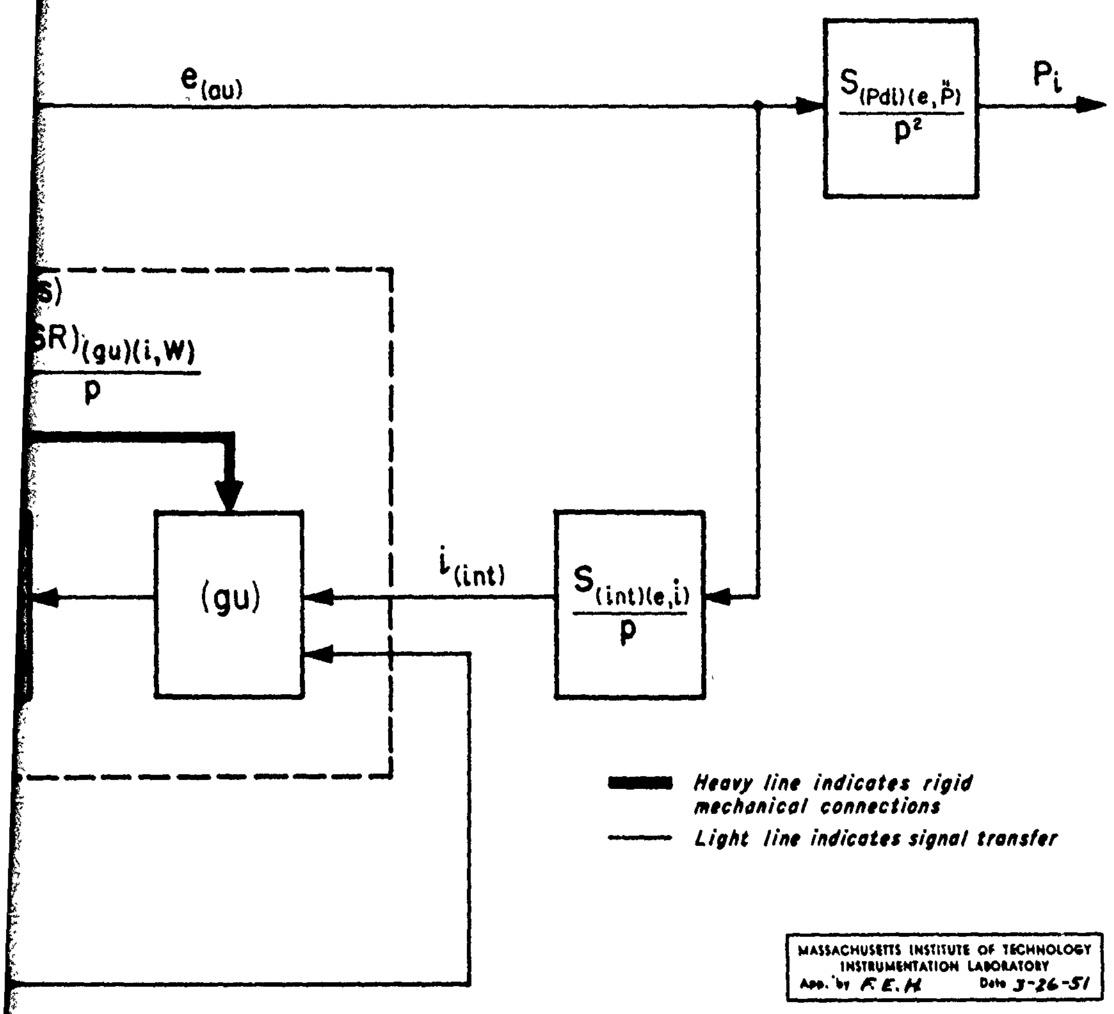


Figure 5-1. Mathematical Functional Diagram of a System for Indicating Position by Means of an Open-Chain Double Integration of the Angular Acceleration of the Indicated Vertical.

DERIVATION 6

POSITION INDICATION BY MEANS OF DIRECT DOUBLE INTEGRATION OF ACCELERATION

Figure 6-1 is a mathematical functional diagram of a system for indicating position by the use of an accelerometer unit which is itself doubly-integrating with respect to time. Its output signal is therefore proportional to indicated position P_i. The indicated position is, from Fig. 6-1:

$$P_i = S_{(ind)(e,P)} \, e_{(diau)} \qquad (6\text{-}1)$$

The accelerometer unit output is, taking into account the double integration action of the accelerometer unit,

$$e_{(diau)} = S_{(diau)(a,\ddot{e})} \frac{1}{p^2} \left(g[(C)V]_{(t,i)} + Rp W_{(EV)t}\right) + \dot{e}_{(diau)o} t + e_{(diau)o} \qquad (6\text{-}2)$$

To perform the indicated double integration in (5-4), it is necessary to evaluate $[(C)V]_{(t,i)}$ for the vertical indicator used here, which is slightly different from that of Fig. 5-1, since it involves the doubly-integrating accelerometer unit. This unit is followed by a differentiator, whose output i_{diff}, proportional to the angular velocity of the indicated vertical, feeds the gyro unit:

$$i_{diff} = S_{diff(\dot{e},i)} \, p \, e_{(diau)} \qquad (6\text{-}3)$$

If the gyro unit is properly compensated for Earth rate,

$$W_{(EV)i} = S_{(cmds)(i,W)} \, i_{diff} \qquad (6\text{-}4)$$

From (3-21) in Derivation 3,

$$p[(C)V]_{(t,i)} = W_{(EV)t} - W_{(EV)i} \qquad [3\text{-}20]$$

Then

$$p[(C)V]_{(t,i)} = W_{(EV)t} - S_{(diau)(a,\ddot{e})}S_{diff(\dot{e},i)}S_{(cmds)(i,W)}\frac{1}{p}(g[(C)V]_{(t,i)}$$

$$+ RpW_{(EV)t}) + \dot{e}_{(diau)o} \qquad (6\text{-}5)$$

The time derivative of (6-5), collecting terms in $[(C)V]_{(t,i)}$, is

$$(p^2 + S_{(diau)(a,\ddot{e})}S_{diff(\dot{e},i)}S_{(cmds)(i,W)}g)\,[(C)V]_{(t,i)}$$

$$= (1 - S_{(diau)(a,\ddot{e})}S_{diff(\dot{e},i)}S_{(cmds)(i,W)}R)pW_{(EV)t} \qquad (6\text{-}6)$$

To minimize the right hand side of (6-6), i.e., to apply Schuler tuning, let

$$S_{(diau)(a,\ddot{e})}S_{diff(\dot{e},i)}S_{(cmds)(i,W)} = \frac{1}{R_s} \qquad (6\text{-}7)$$

and assume $R_s = R$ in (6-6). The result is (3-34), which now implicitly contains (6-7) instead of (3-33) as the Schuler tuning condition.

With this implicit modification, $[(C)V]_{(t,i)}$ is given by

$$[(C)V]_{(t,i)} = [(C)V]_{(t,i)o}\cos\sqrt{\frac{g}{R_s}}\,t + [(\dot{C})V]_{(t,i)o}\sqrt{\frac{R_s}{g}}\sin\sqrt{\frac{g}{R_s}}\,t \qquad (6\text{-}8)$$

which is equivalent to the result (3-35) of Derivation 3.

From eqs. (6-1) and (6-2)

$$P_i = S_{(ind)(e,P)}S_{(diau)(a,\ddot{e})}\frac{1}{p^2}(g[(C)V]_{(t,i)} + RpW_{(EV)t})$$

$$+ S_{(ind)(e,P)}\dot{e}_{(diau)o}t + S_{(ind)(e,P)}e_{(diau)o} \qquad (6\text{-}9)$$

Now adjust the system parameters so that

$$S_{(ind)(e,P)} = \frac{1}{S_{(diau)(a,\ddot{e})}R_s} \qquad (6\text{-}10)$$

Note that

$$(\dot{P}_i)_o = S_{(ind)(e,P)}\,\dot{e}_{(diau)o} \tag{6-11}$$

$$(P_i)_o = S_{(ind)(e,P)}\,e_{(diau)o} \tag{6-12}$$

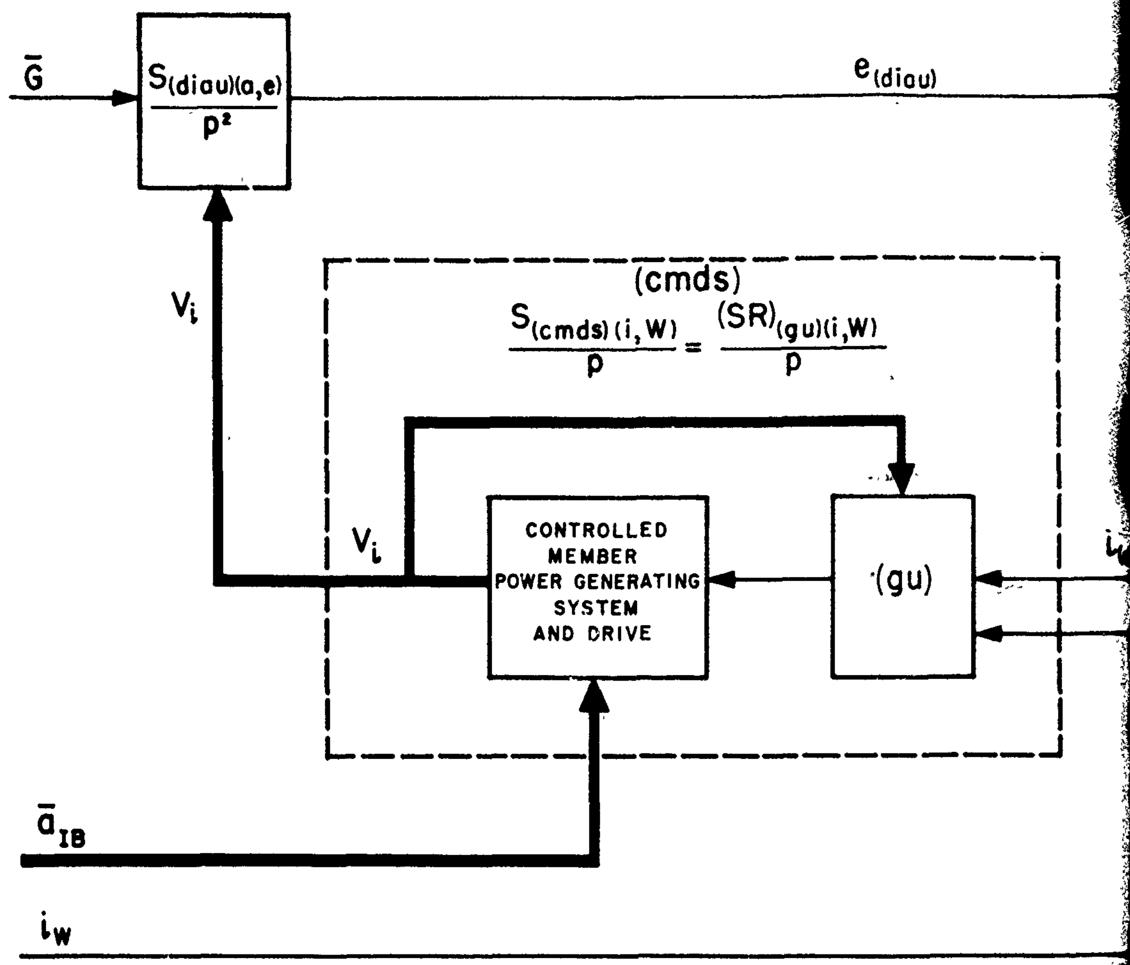
Equation (6-9) becomes

$$P_i = \frac{1}{p^2}\,\frac{g}{R_s}\,[(C)V]_{(t,i)} + \frac{1}{p^2}\,p\,W_{(EV)t} + (\dot{P}_i)_o\,t + (P_i)_o \tag{6-13}$$

This is identical to equation (5-4) of Derivation 5. By carrying out the subsequent operations as given in that derivation, the correction to the indicated position is

$$[(C)P]_{(t,i)} = [(C)P]_{(t,i)o} + [(\dot{C})P]_{(t,i)o}\,t - [(C)V]_{(t,i)o}\left(1 - \cos\sqrt{\frac{g}{R_s}}\,t\right)$$

$$- [(\dot{C})V]_{(t,i)o}\left(t - \sqrt{\frac{R_s}{g}}\,\sin\sqrt{\frac{g}{R_s}}\,t\right) \tag{6-14}$$



G
S(diau)(a,e)
p²
e(diau)
Vi
(cmds)
S(cmds)(i,W) / p = (SR)(gu)(i,W) / p
Vi
CONTROLLED
MEMBER
POWER GENERATING
SYSTEM
AND DRIVE
(gu)
a IB
iW

Fi

2

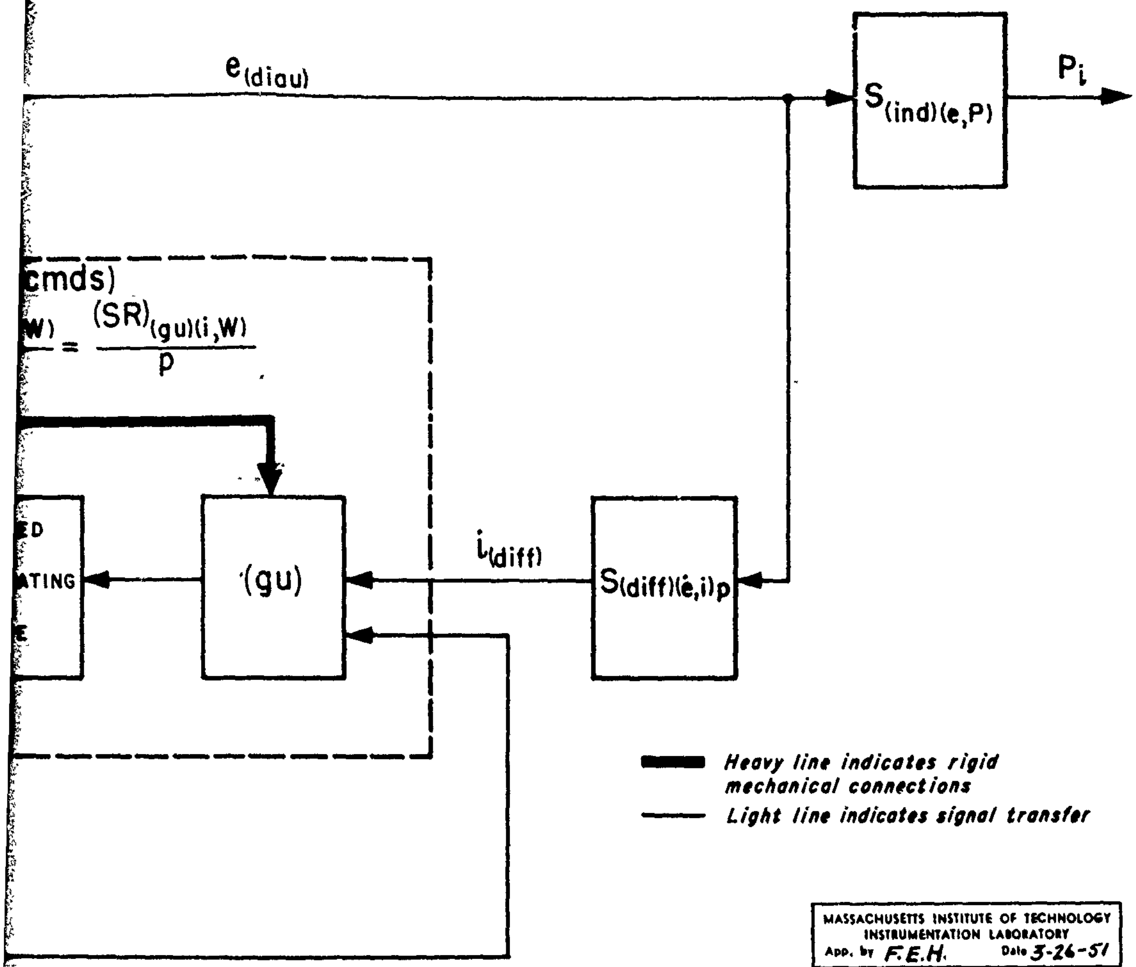


Figure 6-1. Mathematical Functional Diagram of a System for Indicating Position by Means of a Doubly-Integrating Accelerometer Unit.

3

DERIVATION 7

AN AZIMUTH STABILIZATION SYSTEM

Figure 7-1 shows a mathematical functional diagram of a y-axis vertical indicator (similar to Fig. 3-2, Derivation 3) and an azimuth stabilization loop. Let X, Y, Z, be the true position coordinates of the system and let x, y, z, be the indicated position coordinates. Coupling between the y and z vertical indicating loops, ignored in Derivation 3, is a first-order effect here. Note that the Earth-rate correction i_W is not required in the y-axis vertical indicator.

From Fig. 7-1 and Derivation 3

$$e_{(au)y} = S_{(au)(a,e)}\Big[(g[(C)V]_{(t,i)y} + Rp\,W_{[(EV)t]y}\Big] \tag{7-1}$$

$$i_{(int)y} = S_{(int)(e,i)y}\,\frac{1}{p}\,e_{(au)y} + i_{[(int)y]o} \tag{7-2}$$

The torque balance equation for the y-gyro unit is

$$S_{(tg)(i,M)y}\,i_{(int)y} = H\,W_{[(IV)i]y} \tag{7-3}$$

Let

$$S_{(cmds)(i,W)y} = \frac{S_{(tg)(i,M)}}{H} \tag{7-4}$$

The component of the angular velocity of the indicated vertical with respect to inertial space sensed by the y-gyro unit is (neglecting coupling with the x-axis vertical indicating system)

$$W_{[(IV)i]y} = [W_{IE} + (\dot{Lon})]\cos(Lat)\sin[(C)N]_{(t,i)} + W_{[(EV)i]y} \tag{7-5}$$

where $[(C)N]_{(t,i)}$ is defined as the correction to indicated north, i.e., the angle between indicated north and true north; and

(Lat) = latitude

($\dot{Lon}$) = longitude rate

$W_{[(EV)i]y}$ = y-component of the angular velocity of the indicated vertical with respect to the Earth.

Since the function of the azimuth stabilization loop is to minimize $[(C)N]_{(t,i)}$ it is allowable to set

$$\sin\,[(C)N]_{(t,i)} \cong [(C)N]_{(t,i)} \qquad (7\text{-}6)$$

From the definition of $[(C)V]_{(t,i)}$ (Derivation 3B), the y-component of its time derivative is

$$p\,[(C)V]_{(t,i)y} = W_{[(EV)t]y} - W_{[(EV)i]y} \qquad (7\text{-}7)$$

Substituting (7-1),(7-2), (7-3), (7-4), (7-5) into (7-7), and differentiating the result, gives

$$\left[p^2 + S_{(au)(a,e)y}S_{(int)(e,\dot{i})y}S_{(cmds)(i,W)y}\,g\right][(C)V]_{(t,i)y}$$
$$= \left[1 - S_{(au)(a,e)y}S_{(int)(e,\dot{i})y}S_{(cmds)(i,W)y}R\right]pW_{[(EV)t]y}$$
$$+ p\Big[[W_{IE} + (\dot{Lon})]\,[(C)N]_{(t,i)}\cos\,(Lat)\Big] \qquad (7\text{-}8)$$

The Schuler tuning condition is

$$S_{(au)(a,e)y}S_{(int)(e,\dot{i})y}S_{(cmds)(i,W)y} = \frac{1}{R_s} \qquad (7\text{-}9)$$

Equation (7-8) now reduces to

$$(p^2 + \frac{g}{R_s})\,[(C)V]_{(t,i)y} = [W_{IE} + (\dot{Lon})]\,[(C)N]_{(t,i)}\cos\,(Lat) \qquad (7\text{-}10)$$

which is (3-34) except for the non-zero right-hand side created by the aforementioned coupling between y-axis and z-axis systems.

From Fig. (7-1), with $i_{cp} = v_N/S_{(cmds)(i,W)y}R_s$,

$$i_{(mult)z} = \Big\{S_{(int)(i,\dot{i})z}\,\frac{1}{p}\,[i_{(int)y} + \frac{v_N}{S_{(cmds)(i,W)y}R_s}]$$
$$- (i_{(int)x} + i_W)\sin\,(Lat)\Big\}\sec\,(Lat) \qquad (7\text{-}11)$$

where v_N is the northward speed of the ship.

From the torque balance equation at the y-gyro unit (see equations (7-3), (7-4) and (7-5))

$$i_y \equiv i_{(int)y} = \frac{1}{S_{(cmds)(i,W)y}} \left[[(C)N]_{(t,i)} [W_{IE} + (\dot{Lon})] \cos(Lat) + W_{[(EV)i]y} \right] \tag{7-12}$$

From the torque balance equation at the x-gyro unit (neglecting coupling with the y and z systems)

$$i_x \equiv i_{(int)x} + i_W = \frac{1}{S_{(cmds)(i,W)x}} [W_{IE} \cos(Lat) + W_{[(EV)i]x}] \tag{7-13}$$

$$i_z \equiv i_{(mult)z} = \frac{S_{(int)(i,\dot{i})z}}{S_{(cmds)(i,W)y}} \frac{1}{p} \Bigg\{ [(C)N]_{(t,i)} [W_{IE} + (\dot{Lon})] \cos(Lat)$$

$$+ W_{[(EV)i]y} + \frac{v_N}{R_s} \Bigg\} \sec(Lat) - \frac{1}{S_{(cmds)(i,W)x}} [W_{IE} \cos(Lat)$$

$$+ W_{[(EV)i]x}] \tan(Lat) \tag{7-14}$$

From the torque balance equation at the z-gyro unit (neglecting coupling with the x-axis system)

$$i_z = \frac{1}{S_{(cmds)(i,W)z}} \left[- W_{IE} \sin(Lat) + W_{(EN)i} - [(C)V]_{(t,i)y} W_{IE} \cos(Lat) \right] \tag{7-15}$$

or

$$W_{(EN)i} = S_{(cmds)(i,W)z} i_z + W_{IE} \sin(Lat) + [(C)N]_{(t,i)y} W_{IE} \cos(Lat) \tag{7-16}$$

Set

$$S_{(cmds)(i,W)x} = S_{(cmds)(i,W)y} = S_{(cmds)(i,W)z} \tag{7-17}$$

The time rate of change of $[(C)N]_{(t,i)}$ with respect to the base is given by

$$p[\overline{(C)N}]_{(t,i)} = \overline{W}_{(EN)t} - \overline{W}_{(EN)i} - \overline{W}_{EB} \times [\overline{(C)N}]_{(t,i)} \tag{7-18}$$

where $\overline{W}_{EB}$ = angular velocity of the base with respect to the Earth. At marine speeds the cross-product term may be neglected*. Equation (7-18) now becomes

$$p[(C)N]_{(t,i)} = W_{(EN)t} - W_{(EN)i} \qquad (7\text{-}19)$$

Substituting equations (7-14), (7-16), and (7-17) into (7-19)

$$p[(C)N]_{(t,i)} = W_{(EN)t} - S_{(int)(i,\dot{i})z} \frac{1}{p} \Big[[(C)N]_{(t,i)} [W_{IE} + (\dot{Lon})] \cos (Lat)$$

$$+ W_{[(EV)i]y} + \frac{v_N}{R_s} \Big] \sec (Lat) + [W_{IE} \cos (Lat) + W_{[(EV)i]x}] \tan (Lat)$$

$$- W_{IE} \sin (Lat) - [(C)V]_{(t,i)y} W_{IE} \cos (Lat) \qquad (7\text{-}20)$$

From geometrical considerations

$$\left.\begin{aligned} W_{[(EV)t]x} &= (\dot{Lon}) \cos (Lat) \\ W_{(EN)t} &= -\ (\dot{Lon}) \sin (Lat) \\ W_{(EN)t} &= -\ W_{[(EV)t]x} \tan (Lat) \end{aligned}\right\} \qquad (7\text{-}21)$$

$$W_{[(EV)t]y} = -\frac{v_N}{R_s} \qquad (7\text{-}22)$$

$$W_{[(EV)i]y} = W_{[(EV)t]y} - p[(C)V]_{(t,i)y} \qquad (7\text{-}23)$$

Also, assume that

$$W_{[(EV)i]x} = W_{[(EV)t]x} \qquad (7\text{-}24)$$

Substituting the foregoing into (7-20) and differentiating,

---

* Assume the maximum velocity of the ship is 50 knots. The order of magnitude of W_{EB} is then

$$\frac{W_{EB}}{R} = \frac{84.4 \text{ ft/sec}}{20.9 \times 10^6 \text{ ft}} \cong 4 \times 10^{-6} \ \frac{\text{rad}}{\text{sec}}$$

Assuming a correction angle of 10 minutes of arc and the angle between W_{EB} and $[(C)N]_{(t,i)}$ equal to 90 degrees, the magnitude of the cross product term will be of the order of 10^{-9}. This term, then, may be neglected when compared with $W_{(EN)t}$ and $W_{(EN)i}$, each of which has a magnitude of the order of 10^{-6}.

$$\left[p^2 + S_{(int)(i,\dot{i})z}\left[W_{IE} + (\dot{Lon})\right]\right]\left[(C)N\right]_{(t,i)} = \left[S_{(int)(i,\dot{i})z}\sec(Lat)\right]p\left[(C)V\right]_{(t,i)y}$$

$$- p\left\{\left[(C)V\right]_{(t,i)y}\left[W_{IE}\cos(Lat)\right]\right\} \qquad (7\text{-}25)$$

The final term in the above equation is small compared to the preceding term and can be neglected:

$$\left[p^2 + S_{(int)(i,\dot{i})z}\left[W_{IE} + (\dot{Lon})\right]\right]\left[(C)N\right]_{(t,i)} = \left[S_{(int)(i,\dot{i})z}\sec(Lat)\right]p\left[(C)V\right]_{(t,i)y}$$

(7-26)

Since the time variations in $(\dot{Lon})$ and (Lat) are small when compared with the variations in $[(C)N]_{(t,i)}$ and $[(C)V]_{(t,i)y}$, it will be assumed that $(\dot{Lon})$ and (Lat) are quasi-static. This simplifies the subsequent analysis. Then (7-10) and (7-26) combine to give two simultaneous operator equations of the form

$$\left[(p^4 + \frac{g}{R_s}p^2 + S_{(int)(i,\dot{i})z}\left[W_{IE} + (\dot{Lon})\right]\frac{g}{R_s}\right]\left[(C)V\right]_{(t,i)y} = 0 \qquad (7\text{-}27)$$

$$\left[(p^4 + \frac{g}{R_s}p^2 + S_{(int)(i,\dot{i})z}\left[W_{IE} + (\dot{Lon})\right]\frac{g}{R_s}\right]\left[(C)N\right]_{(t,i)} = 0 \qquad (7\text{-}28)$$

The most general solution for either $[(C)V]_{(t,i)y}$ or $[(C)N]_{(t,i)}$ is

$$\left[(C)N\right]_{(t,i)} = C_1 e^{p_1 t} + C_2 e^{p_2 t} + C_3 e^{p_3 t} + C_4 e^{p_4 t} \qquad (7\text{-}29)$$

where C_1, C_2, C_3 and C_4 are constants and p_1, p_2, p_3 and p_4 are the various solutions of

$$p^4 + \frac{g}{R_s}p^2 + S_{(int)(i,\dot{i})z}\left[W_{IE} + (\dot{Lon})\right]\frac{g}{R_s} = 0 \qquad (7\text{-}30)$$

That is,

$$p_{1,2,3,4} = \pm j\sqrt{\frac{g}{2R_s}\left(1 \mp \sqrt{1 - \frac{4S_{(int)(i,\dot{i})z}\left[W_{IE} + (\dot{Lon})\right]R_s}{g}}\right)}$$

(7-31)

where $j = \sqrt{-1}$

The system will be stable (i.e., oscillatory with limited amplitude) only if the quantity under the inner radical is positive; i.e., if

$$\frac{4S_{(int)(i,\dot{i})z}[W_{IE} + (\dot{Lon})]R_s}{g} \leq 1 \qquad (7\text{-}32)$$

or

$$S_{(int)(i,\dot{i})z} \leq \frac{g}{4[W_{IE} + (\dot{Lon})]R_s} \qquad (7\text{-}33)$$

Assume that this condition is met by suitable adjustment of $S_{(int)(i,\dot{i})z}$. The complete solution of (7-28) (that for (7-29) is similar) is then of the form:

$$[(C)N]_{(t,i)} = C_1 \cos W_{n_1} t + C_2 \sin W_{n_1} t + C_3 \cos W_{n_2} t + C_4 \sin W_{n_2} t \qquad (7\text{-}34)$$

where C_1, C_2, C_3 and C_4 are constants to be determined by the initial conditions, and W_{n_1} and W_{n_2} are given by

$$W_{n_1} = \sqrt{\frac{g}{2R_s}\left(1 - \sqrt{1 - \frac{4S_{(int)(i,\dot{i})z}[W_{IE} + (\dot{Lon})]R_s}{g}}\right)} \qquad (7\text{-}35)$$

$$W_{n_2} = \sqrt{\frac{g}{2R_s}\left(1 + \sqrt{1 - \frac{4S_{(int)(i,\dot{i})z}[W_{IE} + (\dot{Lon})]R_s}{g}}\right)}$$

Let the initial conditions be represented by $([(C)V]_{(t,i)})_o$, $([(\dot{C})V]_{(t,i)})_o$, $([(C)N]_{(t,i)})_o$, $([(\dot{C})N]_{(t,i)})_o$, the y-component of the correction to the indicated vertical and its first derivative, and the correction to indicated north and its first derivative, respectively, at the start of the problem. The constants C_1, C_2, C_3 and C_4 can be given in terms of the initial conditions and W_{n_1} and W_{n_2}. These constants, with (7-34) and (7-35), then give the performance equation. The system is doubly periodic, with angular frequencies W_{n_1} and W_{n_2}. A similar result, with the same initial conditions, holds for the y-system.

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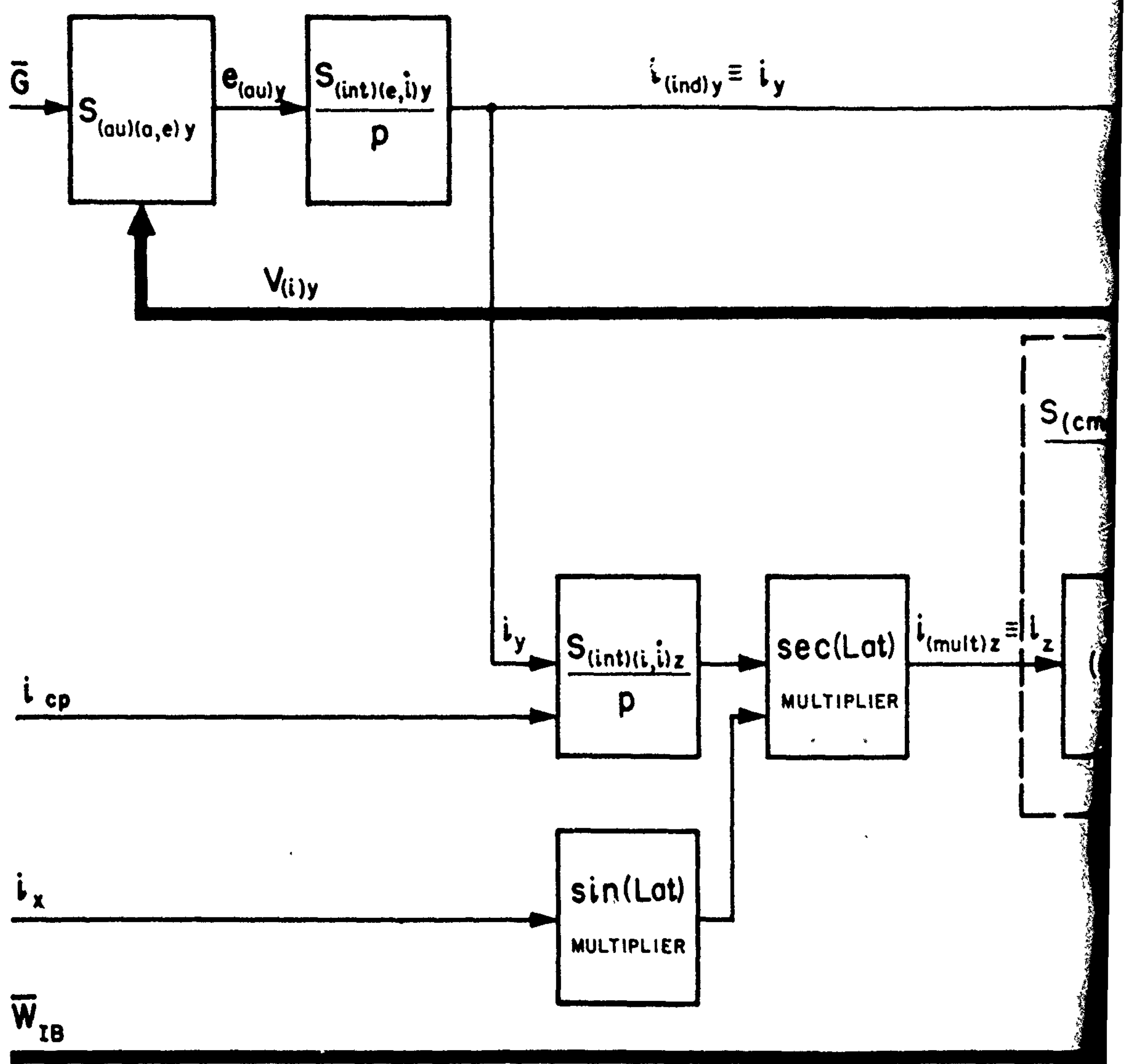

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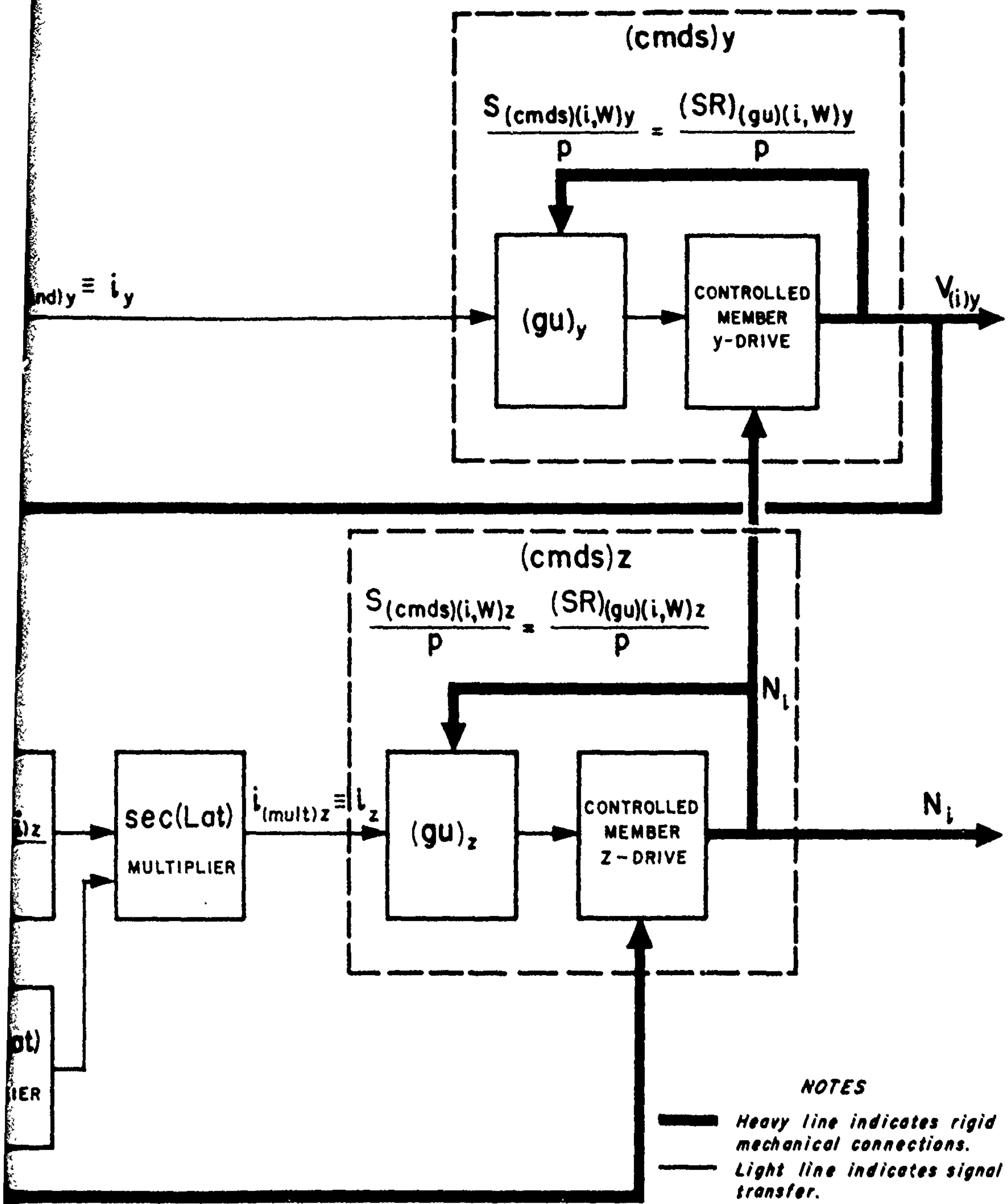


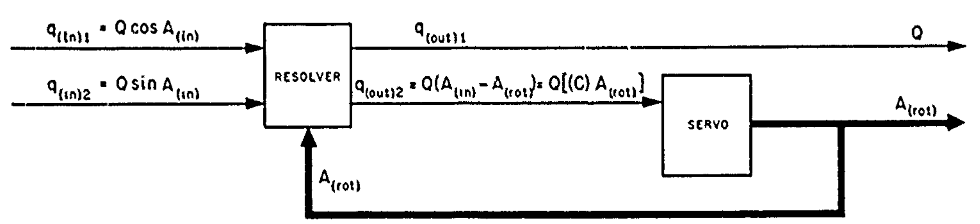
Figure 7-1. Mathematical Functional Diagram of Combined Y-Axis Vertical Indicator and Z-Axis Stabilizer.

DERIVATION 8

POSITION INDICATION BY MEANS OF CELESTIAL LONGITUDE SIGNAL COMPOSITION RATE

Figure 8-2 shows the three-axis stabilization system resulting from combination of the systems of Fig. 3-2 in Derivation 3 and Fig. 7-1 in Derivation 7. Base motion isolation is also provided here by the roll, pitch and azimuth drives at the right of Fig. 8-2. The principal outputs of this system are latitude and longitude, derived basically by the resolver in the lower right of Fig. 8-2, which is changed from its usual function to act here as a vector-component compositor (Fig. 8-1). Equations (7-15) and (7-21) in Derivation 7 give the compositor inputs:

$$
\begin{aligned}
i_x \equiv i_{(int)x} + i_W = \frac{1}{S_{(cmds)(i,W)x}} \Big\{ & [W_{IE} + (\dot{Lon})] \cos (Lat) \\
& - [(C)V]_{(t,i)y} [W_{IE} + (\dot{Lon})] \sin (Lat) \\
& + (\dot{Lat}) [(C)N]_{(t,i)} - p[(C)V]_{(t,i)x} \Big\}
\end{aligned}
\qquad (8\text{-}1)
$$



If the resolver is used as a vector compositor

$q_{(in)1} = Q \cos A_{(in)}$ $\qquad q_{(out)1} = q_{(in)1} \cos A_{(rot)} + q_{(in)2} \sin A_{(rot)}$

$q_{(in)2} = Q \sin A_{(in)}$ $\qquad q_{(out)2} = q_{(in)2} \cos A_{(rot)} - q_{(in)1} \sin A_{(rot)}$

$q_{(out)1} = Q (\cos A_{(in)} \cos A_{(rot)} + \sin A_{(in)} \sin A_{(rot)}) = Q \cos (A_{(in)} - A_{(rot)}) \cong Q$

$q_{(out)2} = Q (\sin A_{(in)} \cos A_{(rot)} - \cos A_{(in)} \sin A_{(rot)}) = Q \sin (A_{(in)} - A_{(rot)}) \cong Q[(C)A_{(rot)}]$

Figure 8-1. Resolver Used as a Vector Compositor.

$$i_z \equiv i_{(mult)z} = \frac{1}{S_{(cmds)(i,W)z}} \Big\{ - [W_{IE} + (\dot{Lon})] \sin (Lat) - (\dot{Lat})[(C)V]_{(t,i)x}$$

$$- [(C)V]_{(t,i)y} [W_{IE} + (\dot{Lon})] \cos (Lat) - p[(C)N]_{(t,i)} \Big\} \qquad (8\text{-}2)$$

Under equilibrium settled conditions,

$$[(C)V]_{(t,i)x} = [(C)V]_{(t,i)y} = [(C)N]_{(t,i)} = 0 \qquad (8\text{-}3)$$

The following correspondence can then be set up between the generalized vector compositor inputs of Fig. 8-1 and the compositor input signals of equations (8-1) and (8-2).

$$A_{(in)} \cong A_{(rot)} = (Lat)_i = A_{(comp)(rot)(out)} \qquad (8\text{-}4)$$

and if sensitivities are adjusted so that

$$S_{(cmds)(i,W)x} = S_{(cmds)(i,W)z} \qquad (8\text{-}5)$$

it follows that

$$Q = \frac{1}{S_{(cmds)(i,W)z}} [W_{IE} + (\dot{Lon})] \qquad (8\text{-}6)$$

Thus, the inputs are

$$q_{(in)1} = \frac{1}{S_{(cmds)(i,W)x}} [W_{IE} + (\dot{Lon})] \cos (Lat) = i_x \qquad (8\text{-}7)$$

and

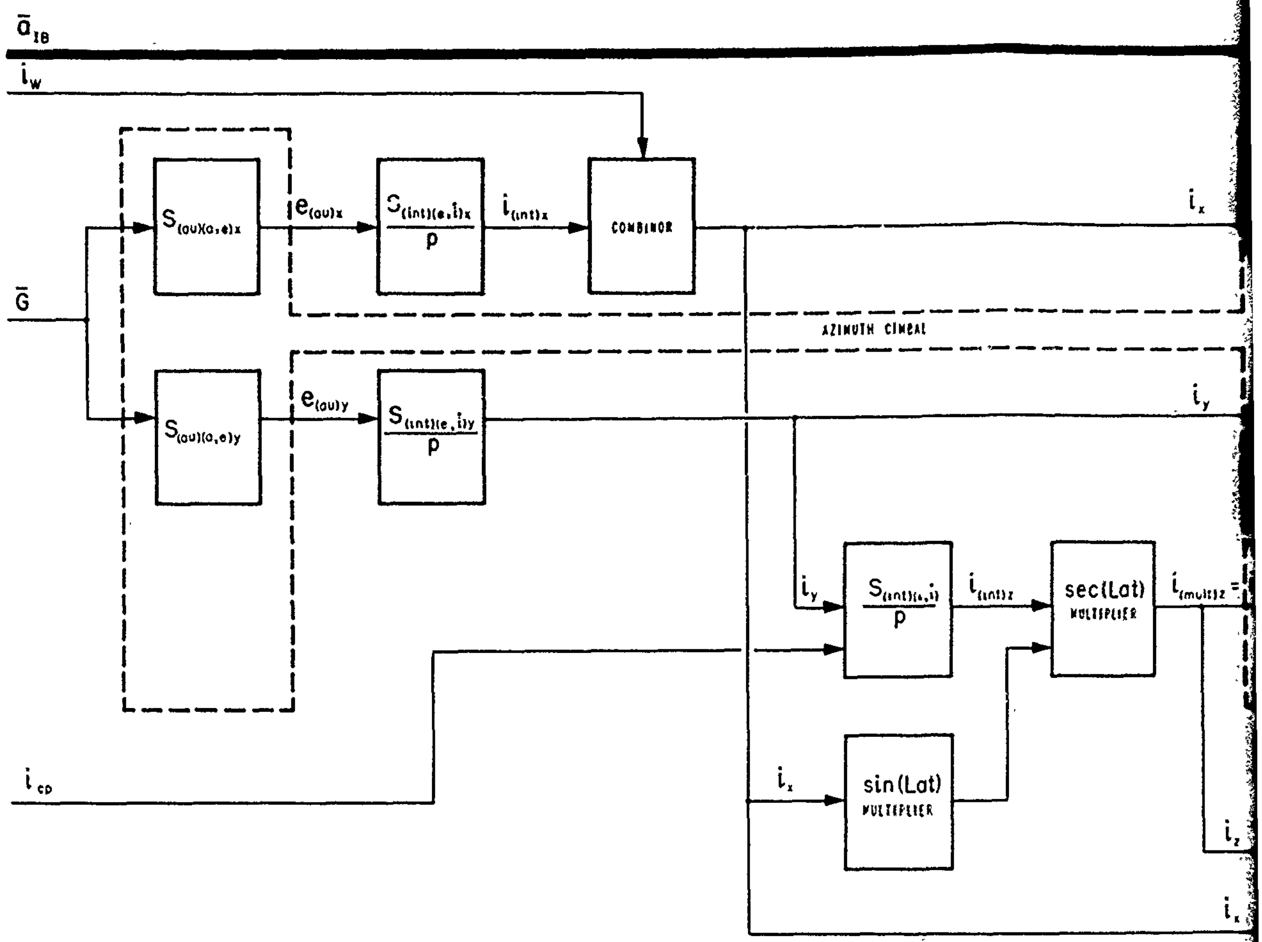
$$q_{(in)2} = \frac{1}{S_{(cmds)(i,W)z}} [W_{IE} + (\dot{Lon})] \sin (Lat) = - i_z \qquad (8\text{-}8)$$

In Fig. 8-1, the sensitivity of the compositor for a dual current input and voltage output is assumed to be unity. In the general case, if this sensitivity is $S_{(comp)(i_1,i_2;e)}$, the compositor output is

$$q_{(out)_1} = e_{(comp)(out)} \cong \frac{S_{(comp)(i_1,i_2;e)}}{S_{(cmds)(i,W)z}} [W_{IE} + (\dot{Lon})] \qquad (8\text{-}9)$$

If (8-3) is not satisfied, a sinusoidal variation, constituting an inaccuracy in indication, is superimposed on $e_{(comp)(out)}$ and $A_{(comp)(rot)(out)}$. A discussion of the effects of this condition will be deferred here and taken up in the next report.

The component marked "clock" in Fig. 8-2 furnishes a signal $e_{(W_{IE})}$ proportional to W_{IE}. The output of the differential, which receives $e_{(comp)(out)}$ and $e_{(W_{IE})}$ as inputs, is proportional to $(\dot{Lon})$. This quantity is then integrated. The integrator output is proportional to the longitude change. A longitude reference signal $e_{(Lon)(ref)}$ is then added to this to obtain the angular difference between indicated longitude and the reference.



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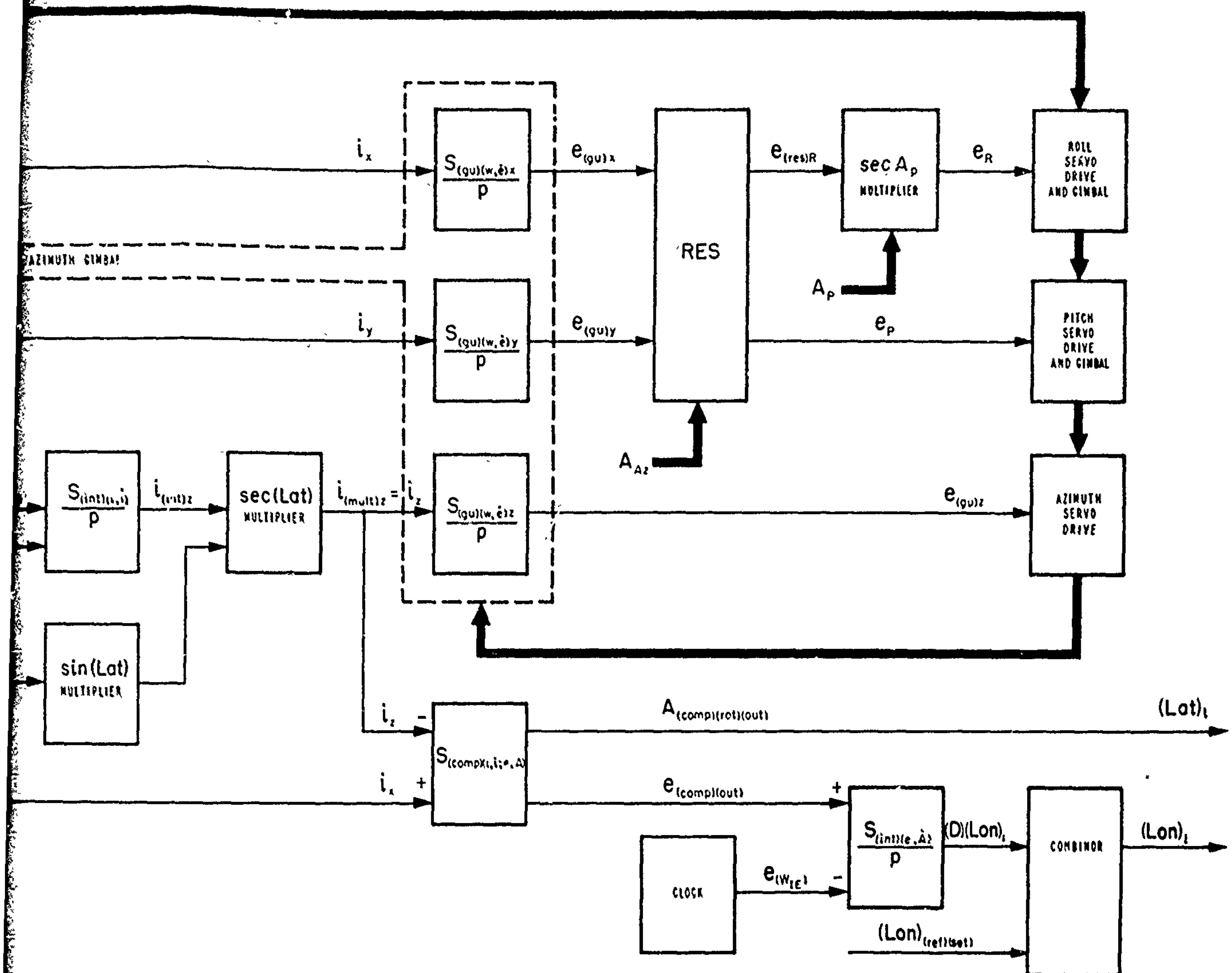


Figure 8-2. Mathematical Functional Diagram of a Three-Axis Position Indicator Using Vector Composition of the Indicated Vertical Angular Velocity Components.

DERIVATION 9

INDICATION OF POSITION BY THE DETECTION OF CELESTIAL LONGITUDE RATE FROM A STABLE PLATFORM

A. Latitude System

Figure 9-1 shows a mathematical functional diagram of this system. In the diagram, a single-axis vertical indicator is shown; actually, a completely stabilized vertical and azimuth indicator (as in Fig. 8-2) would be required, with its controlled member supporting the latitude gimbal of Fig. 9-1. This gimbal supports the latitude gyro unit, which, with its accompanying servo, tracks a direction perpendicular to the Earth's polar axis. The indicated polar axis will be defined as $(PA)_i$, the true polar axis as $(PA)_t$, and the correction to the indicated polar axis as:

$$[\overline{(C)PA}]_{(t,i)} \equiv \overline{1}_{(PA)_i} \times \overline{1}_{(PA)_t} \tag{9-1}$$

(See the lower part of Fig. 9-1.) The torque summation equation for the latitude gyro unit is:

$$H\Big[[W_{IE} + (\dot{Lon})] [(C)PA]_{(t,i)} + p[(C)V]_{(t,i)x} \sin (Lat)_i + (\dot{Lat})_t [(C)N]_{(t,i)} \sin (Lat)_i + p[(C)N]_{(t,i)} \cos (Lat)_i \Big] = S_{(er)(A,M)} A_g \tag{9-2}$$

where $S_{(er)(A,M)}$ is the elastic restraint sensitivity of the gyro unit and A_g is the gyro unit gimbal angle*. The microsyn signal generator on the gyro rotor shaft produces an output signal proportional to the rotor angle. This signal is then integrated to give

$$i_{(Lat)} = S_{(sg)(A,e)} S_{(int)(e,\dot{i})} \frac{1}{p} A_g + (i_{(Lat)})_o \tag{9-3}$$

$$p\,(Lat)_i = S_{(Lds)(i,\dot{A})}\, i_{(Lat)} \tag{9-4}$$

---

* Since the dynamic response of the gyro unit is much faster than the response of the tracking loop, the inertia and damping terms are neglected.

From the geometric sketch in Fig. 9-1,

$$(\mathrm{Lat})_i = (\mathrm{Lat})_t + [(C)PA]_{(t,i)} - [(C)V]_{(t,i)y} \qquad (9\text{-}5)$$

From (9-3),

$$p\,(\mathrm{Lat})_i = S_{(\mathrm{Lds})(i,\dot{A})} S_{(\mathrm{int})(e,\dot{i})} S_{(\mathrm{sg})(A,e)} \frac{1}{p} A_g + S_{(\mathrm{Lds})(i,\dot{A})}\, {}^{i}(\mathrm{Lat})_o \qquad (9\text{-}6)$$

Substituting (9-2) into (9-6) and differentiating:

$$p^2 (\mathrm{Lat})_i = \frac{H\, S_{(\mathrm{sg})(A,e)} S_{(\mathrm{int})(e,\dot{i})} S_{(\mathrm{Lds})(i,\dot{A})}}{S_{(\mathrm{er})(A,M)}} \Big[[W_{IE} + (\dot{\mathrm{Lon}})][(C)PA]_{(t,i)} + \left\{ p\,[(C)V]_{(t,i)x} + (\dot{\mathrm{Lat}})_t [(C)N]_{(t,i)} \right\} \sin(\mathrm{Lat})_i + p\,[(C)N]_{(t,i)} \cos(\mathrm{Lat})_i \Big] \qquad (9\text{-}7)$$

Define the correction to the indicated latitude:

$$[(C)(\mathrm{Lat})]_{(t,i)} \equiv (\mathrm{Lat})_t - (\mathrm{Lat})_i \qquad (9\text{-}8)$$

From (9-5)

$$\begin{aligned} [(C)PA]_{(t,i)} &= (\mathrm{Lat})_i - (\mathrm{Lat})_t + [(C)V]_{(t,i)y} \\ &= -[(C)(\mathrm{Lat})]_{(t,i)} + [(C)V]_{(t,i)y} \end{aligned} \qquad (9\text{-}9)$$

Using (9-8) and (9-9), (9-7) can be written as follows:

$$\begin{aligned} -\left[p^2 + \frac{H\, S_{(\mathrm{sg})(A,e)} S_{(\mathrm{int})(e,\dot{i})} S_{(\mathrm{Lds})(i,\dot{A})}}{S_{(\mathrm{er})(A,M)}} [W_{IE} + (\dot{\mathrm{Lon}})]\right] [(C)(\mathrm{Lat})]_{(t,i)} \\ = -(\ddot{\mathrm{Lat}})_t + \frac{H\, S_{(\mathrm{sg})(A,e)} S_{(\mathrm{int})(e,\dot{i})} S_{(\mathrm{Lds})(i,\dot{A})}}{S_{(\mathrm{er})(A,M)}} \Big[\Big\{ p\,[(C)V]_{(t,i)x} \\ + (\dot{\mathrm{Lat}})_t [(C)N]_{(t,i)} \Big\} \sin(\mathrm{Lat})_i + p\,[(C)N]_{(t,i)} \cos(\mathrm{Lat})_i \\ + [W_{IE} + (\dot{\mathrm{Lon}})]\, [(C)V]_{(t,i)y} \Big] \end{aligned} \qquad (9\text{-}10)$$

When the vertical indicator is in a settled equilibrium state, so that, approximately,

$$[(C)V]_{(t,i)x} = [(C)V]_{(t,i)y} = [(C)N]_{(t,i)} = 0 \qquad (9\text{-}11)$$

the second term on the left-hand side of (9-10) is zero. The solution of (9-10) is then, assuming a constant acceleration in latitude:

$$[(C)(Lat)]_{(t,i)} = \left\{[(C)(Lat)]_{(t,i)}\right\}_o \cos W_{nL}t + \frac{1}{W_{nL}}\left\{[(C)(\dot{Lat})]_{(t,i)}\right\}_o \sin W_{nL}t$$

$$+ \frac{(\ddot{Lat})}{W^2_{nL}} \qquad (9\text{-}12)$$

where

$$W_{nL} = \sqrt{\frac{HS_{(sg)(A,e)}S_{(int)(e,i)}S_{(Lds)(i,A)}}{S_{(er)(A,M)}}[W_{IE} + (\dot{Lon})]} \qquad (9\text{-}13)$$

B. Longitude System

The torque summation equation for the longitude gyro unit is:

$$H\left\{[W_{IE} + (\dot{Lon})_t] \cos [(C)PA]_{(t,i)} + (p[(C)V]_{(t,i)x}) \cos (Lat)_i\right.$$

$$+ (\dot{Lat})_t [(C)N]_{(t,i)} \cos (Lat)_i - (p[(C)N]_{(t,i)}) \sin (Lat)_i + W_{tds}$$

$$\left. + W_{lds}\right\} = cpA_g \qquad (9\text{-}14)$$

where $W_{tds} = -W_{(IE)i}$, the indicated angular velocity of the sidereal time drive system, and $W_{lds} = -(\dot{Lon})_i$, the angular velocity of the longitude gimbal drive system with respect to the base. (In this analysis $W_{(IE)i}$ will be assumed equal to W_{IE}). $[(C)PA]_{(t,i)}$ and $[(C)N]_{(t,i)}$ are, as before, assumed to be small angles, nearly equal to their respective sines.

The gyro unit signal generator output is:

$$e_{(sg)} = S_{(sg)(A,e)}A_g \qquad (9\text{-}15)$$

From (9-15) and the definitions above,

$$-(\dot{Lon})_i = W_{lds} = -S_{(lds)(e,W)}\, e_{sg} \tag{9-16}$$

which makes (9-14):

$$(\dot{Lon})_t - (\dot{Lon})_i + (p[(C)V]_{(t,i)x}) \cos (Lat)_i + (\dot{Lat})_t [(C)N]_{(t,i)} \cos (Lat)_i$$

$$- (p[(C)N]_{(t,i)}) \sin (Lat)_i = \frac{c}{HS_{(lds)(e,W)} S_{(sg)(A,e)}} (\ddot{Lon})_i \tag{9-17}$$

Define

$$\left.\begin{aligned}
&[(C)(Lon)]_{(t,i)} \equiv (Lon)_t - (Lon)_i \\
&(D)(Lon)_t \equiv (Lon)_t - (Lon)_{(t)o} \\
&(D)(Lon)_i \equiv (Lon)_i - (Lon)_{(i)o} \\
&\text{where} \\
&(Lon)_t = (D)(Lon)_t + (Lon)_{(ref)t} \\
&\text{and} \\
&(Lon)_i = (D)(Lon)_i + (Lon)_{(ref)t} - [(C)(Lon)]_{(ref)}
\end{aligned}\right\} \tag{9-18}$$

where $[(C)(Lon)]_{(ref)}$ is the correction to the indicated reference longitude setting. Therefore,

$$p[(C)(Lon)]_{(t,i)} = (D)(\dot{Lon})_t - (D)(\dot{Lon})_i = (\dot{Lon})_t - (\dot{Lon})_i \tag{9-19}$$

From (9-17) and (9-19),

$$p\left(1 + \frac{cp}{HS_{(lds)(e,W)} S_{(sg)(A,e)}}\right) [(C)(Lon)]_{(t,i)} = \frac{c}{HS_{(lds)(e,W)} S_{(sg)(A,e)}} (\ddot{Lon})_t$$

$$- (p[(C)V]_{(t,i)x}) \cos (Lat)_i - (\dot{Lat})_t [(C)N]_{(t,i)} \cos (Lat)_i$$

$$+ (p[(C)N]_{(t,i)}) \sin (Lat)_i \tag{9-20}$$

When the controlled member is in equilibrium,

$$p[(C)V]_{(t,i)x} = p[(C)N]_{(t,i)} = [(C)N]_{(t,i)} = 0 \tag{9-21}$$

This reduces (9-20) to:

$$(p + (CT)_{lds} p^2)\,[(C)(Lon)]_{(t,i)} = (CT)_{lds}\,(\ddot{Lon})_t \tag{9-22}$$

where

$$(CT)_{lds} \equiv \frac{c}{HS_{(lds)(e,W)}\,S_{(sg)(A,e)}} \tag{9-23}$$

The most general solution of (9-22) is

$$p\,[(C)(Lon)]_t = e^{-\frac{t}{(CT)_{lds}}} \left\{ \frac{1}{p}\,[(\ddot{Lon})_t\, e^{\frac{t}{(CT)_{lds}}}] + [(\dot{Lon})_{(t)o} - (\dot{Lon})_{(i)o}] \right\} \tag{9-24}$$

Two cases are considered: the longitude rate may be constant or changing. If it is constant,

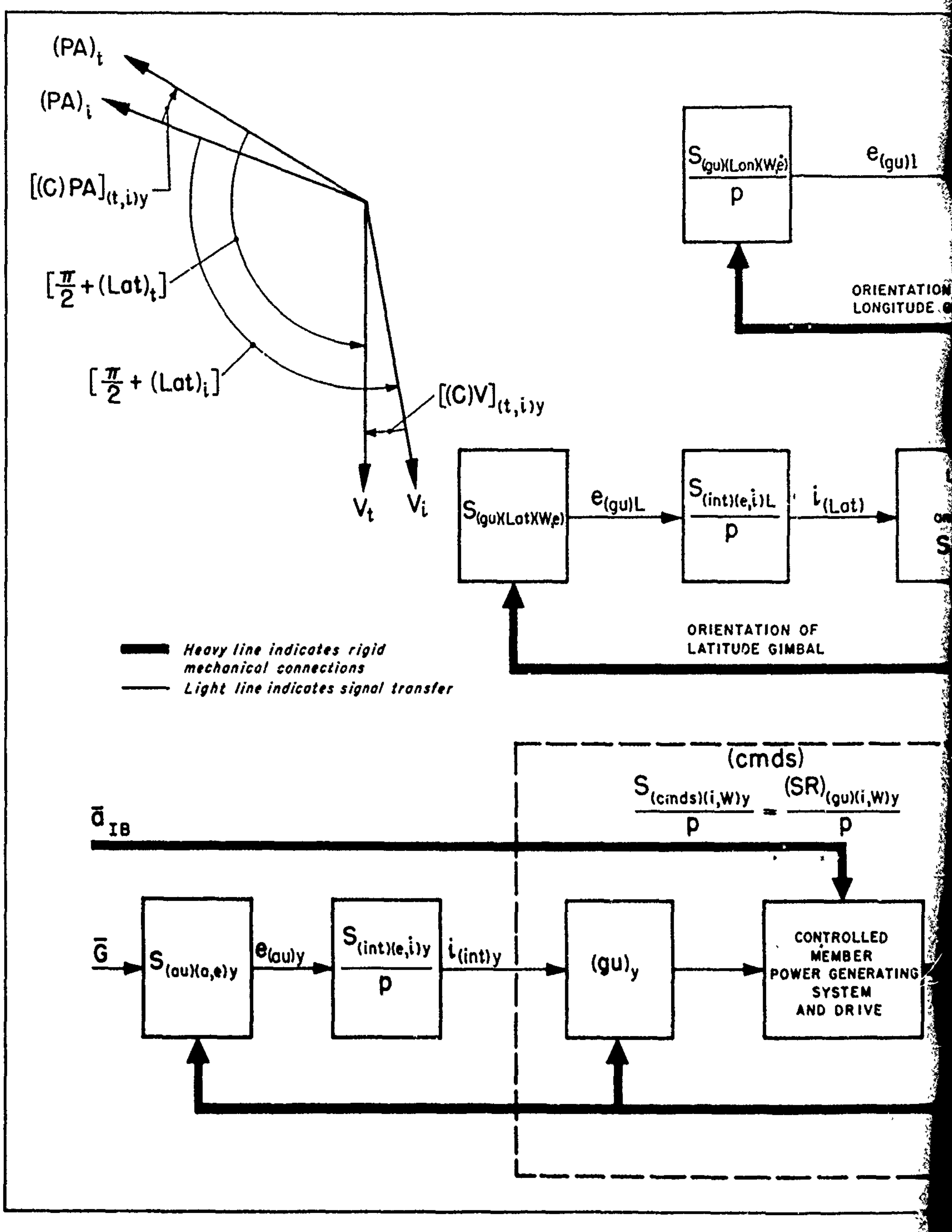
$$\left.\begin{aligned} (\ddot{Lon})_t &= 0 \\ (\dot{Lon})_t &= [(\dot{Lon})_t]_o \end{aligned}\right\} \tag{9-25}$$

$$[(C)(Lon)]_{(t,i)} = [(C)(Lon)]_{(ref)} + (CT)_{lds}[(Lon)_{(t)o} - (Lon)_{(i)o}][1 - e^{\frac{t}{(CT)_{lds}}}] \tag{9-26}$$

In (9-26), $[(C)(Lon)]_{(ref)}$ is the correction to the initial reference setting in longitude. The term in $(CT)_{lds}$ is such that a large loop sensitivity ($(CT)_{lds}$ small) results in a highly damped system as far as the initial velocity lag ($[(\dot{Lon})_t]_o - [(\dot{Lon})_i]_o$) is concerned. When $(CT)_{lds}$ is sufficiently small, the correction to the indicated longitude reduces to

$$[(C)(Lon)]_{(t,i)} = [(C)(Lon)]_{(ref)} \tag{9-27}$$

The presence of an acceleration in longitude negates (9-24), and the velocity lag is then a function of the time. The lag may be positive or negative, depending on the sign of the acceleration.



(PA)t
(PA)i
[(C)PA](t,i)y
[π/2 + (Lat)t]
[π/2 + (Lat)i]
[(C)V](t,i)y
Vt
Vi
S(gu)(Lon)(Wie) / p
e(gu)l
ORIENTATION
LONGITUDE
S(gu)(Lat)(Wie)
e(gu)L
S(int)(e,i)L / p
i(Lat)
ORIENTATION OF
LATITUDE GIMBAL
Heavy line indicates rigid mechanical connections
Light line indicates signal transfer
(cmds)
S(cmds)(i,W)y / p = (SR)(gu)(i,W)y / p
a IB
G
S(au)(a,e)y
e(au)y
S(int)(e,i)y / p
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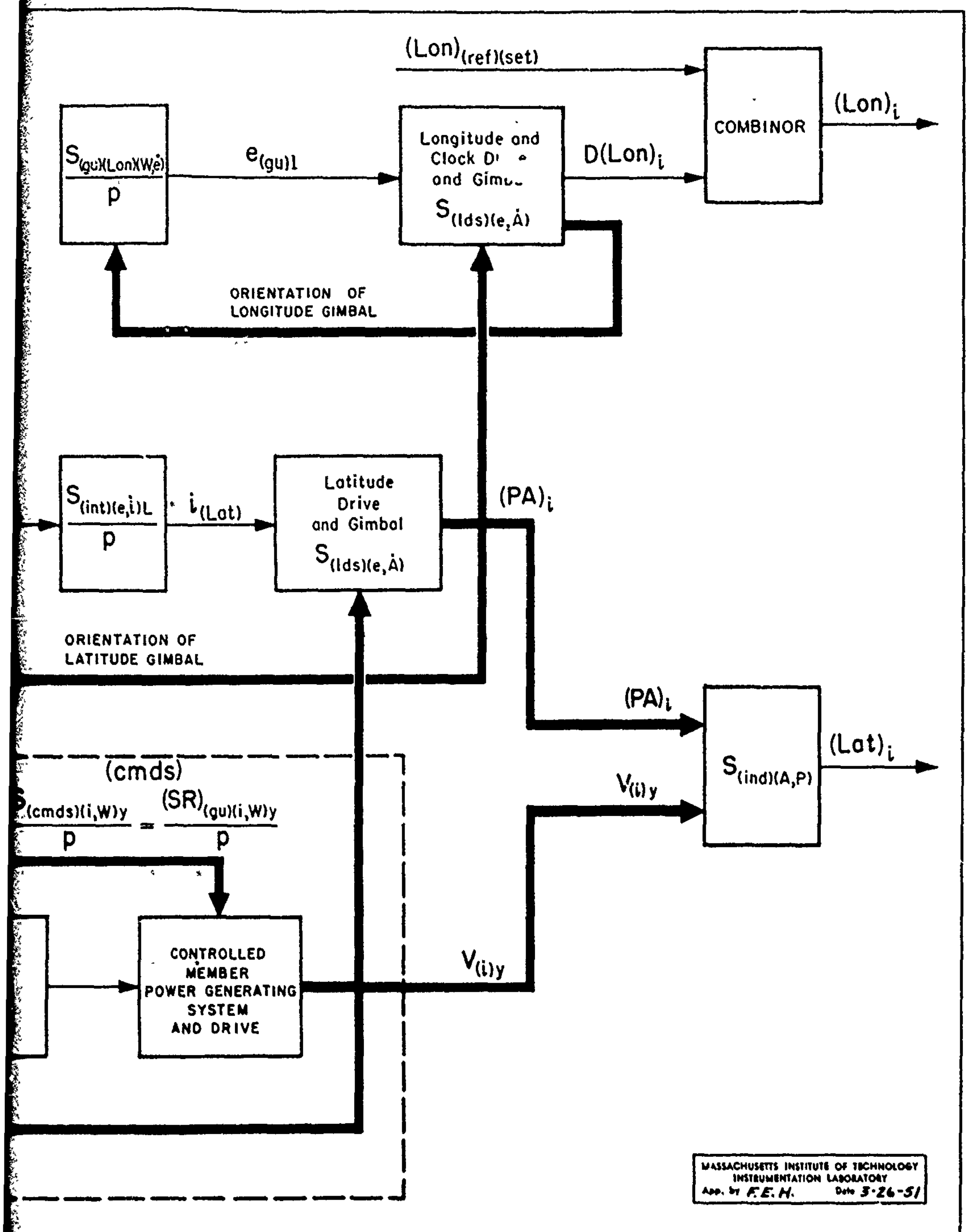

Figure 9-1. Mathematical Functional Diagram of a Position Indicator Using the Detection of Earth-Rate from a Stable Platform.

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DERIVATION 10

INDICATION OF POSITION BY COMPARING INDICATED AND REFERENCE VERTICALS, USING A PRE-ALIGNED INERTIAL GYRO UNIT

A. Latitude Indication

Figure 10-2 shows a mathematical functional diagram for this system. The function of the controlled member drive system here is the same as in Derivation 3, and (3-35) is applicable.

Refer to Fig. 10-1. Define

$$[(C)(Lat)]_{(t,i)} = (Lat)_t - (Lat)_i \qquad (10\text{-}1)$$

From the figure,

$$(Lat)_i = (Lat)_t + [(C)V]_{(t,i)y} - [(C)PA]_{(t,i)y} \qquad (10\text{-}2)$$

Therefore

$$[(C)(Lat)]_{(t,i)} = [(C)PA]_{(t,i)y} - [(C)V]_{(t,i)y} \qquad (10\text{-}3)$$

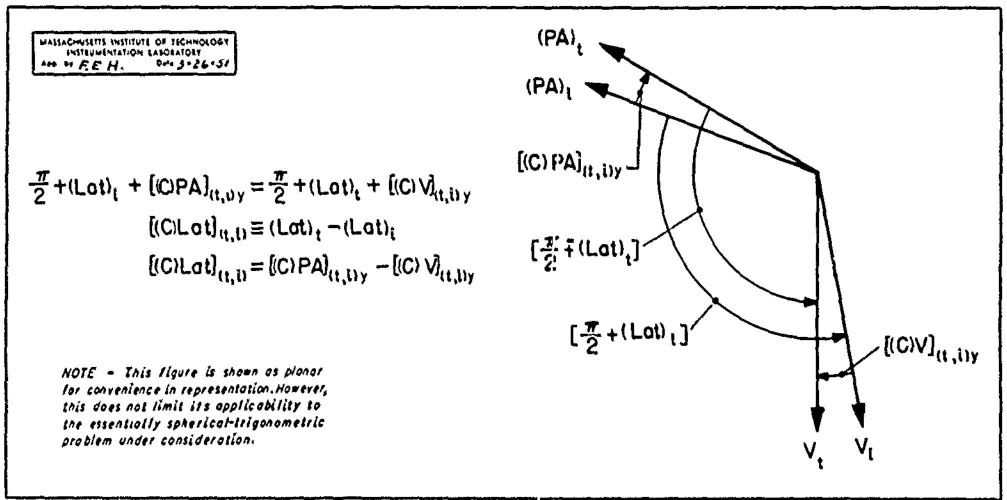


Figure 10-1. Corrections to the Indicated Polar Axis and Indicated Vertical Pertinent to Latitude Indication.

Applying (3-35) to $[(C)V]_{(t,i)y}$ in (10-3)

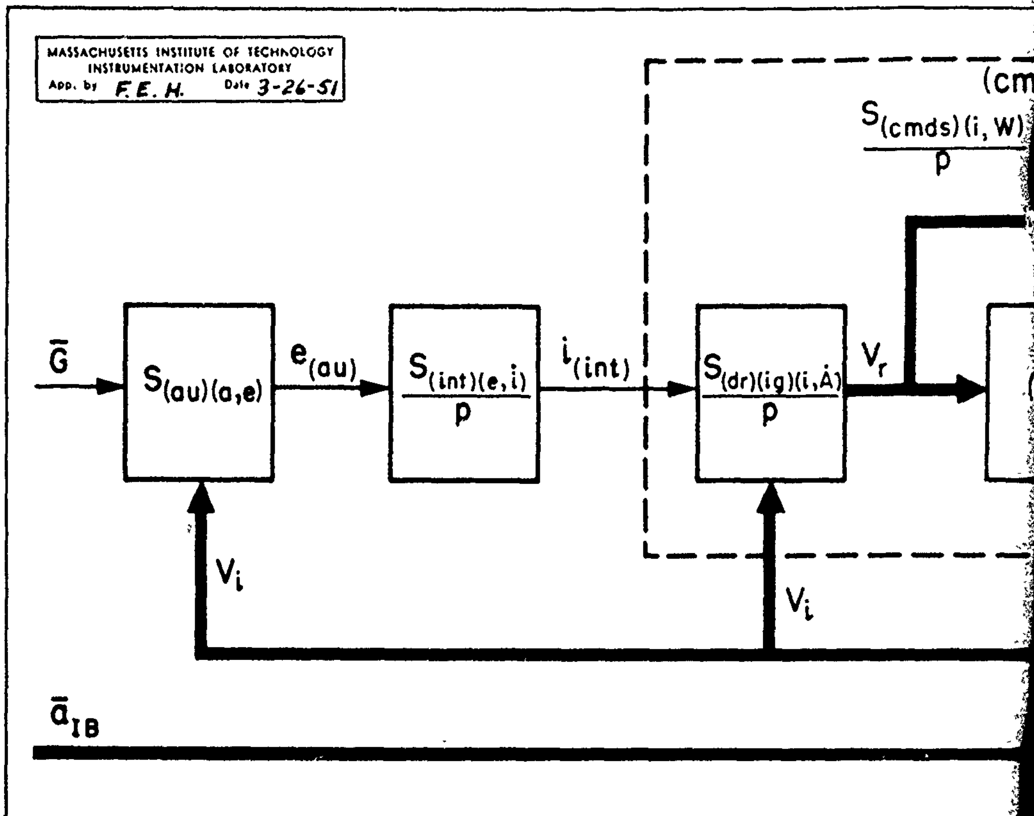
$$[(C)(Lat)]_{(t,i)} = [(C)PA]_{(t,i)y} - ([(C)V]_{(t,i)y})_o \cos\sqrt{\frac{g}{R_s}}\, t$$

$$-\sqrt{\frac{R_s}{g}}\, ([(C)V]_{(t,i)y})_o \sin \frac{g}{R_s}\, t \qquad (10\text{-}4)$$

Here, $[(C)PA]_{(t,i)y}$ represents the meridian plane component of a correction to the indicated reference vertical due to a possible misalignment of the inertial gyro units with respect to the Earth's true polar axis.

B. Longitude Indication

Refer to part B of Derivation 9. The case here is similar, except for the character of the reference line from which longitude difference is measured; in the former case, this line is maintained by the combined action of the controlled member stabilization and the orientation of the latitude gimbal, while in the present case, the line is maintained by the orientation of the inertial gyros mounted on the inertial space gimbal, and their associated drives. Note, however, that the present system requires an initial alignment of the inertial gyros, while that of Derivation 9B is self-settling and requires no such alignment.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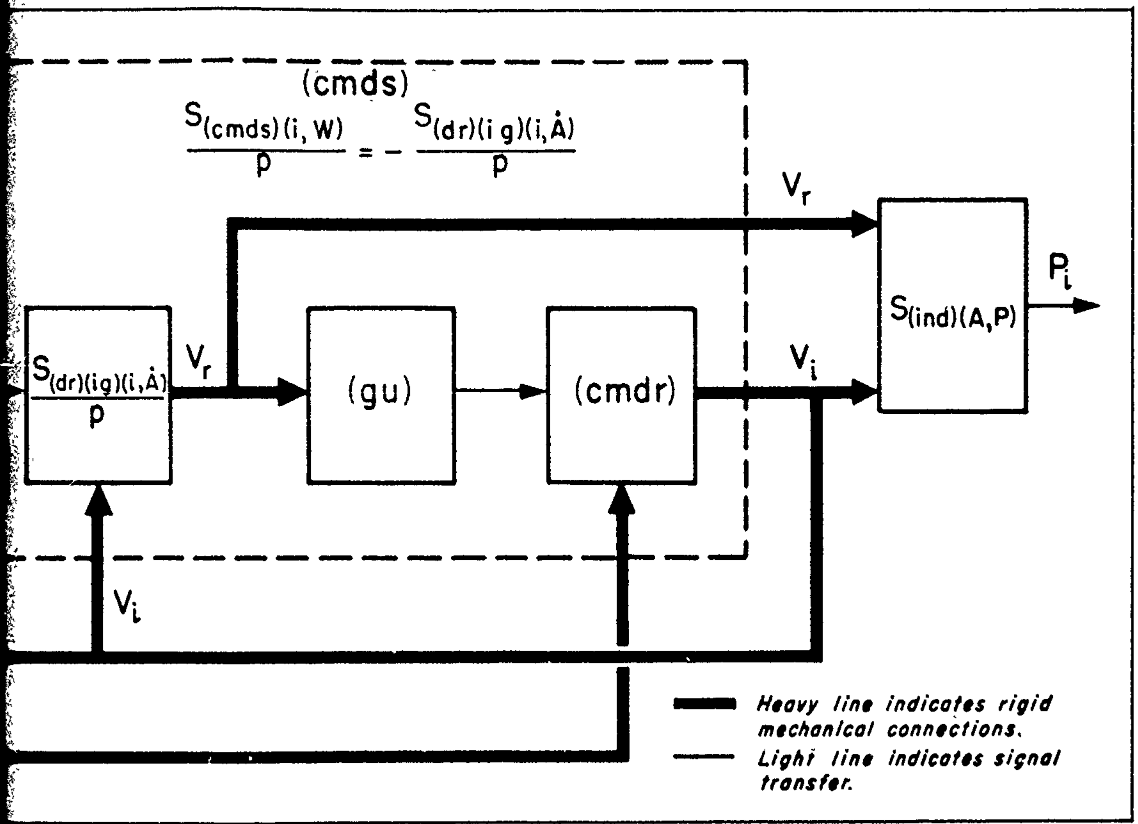

Figure 10-2. Mathematical Functional Diagram of a Single-Axis Position Indicator Using an Inertial Reference Vertical.

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